①

ELASTIC ANALYSIS OF A CIRCULAR TOROIDAL SHELL

USING COMPUTER MODELING

by

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1

88 11   101

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ABSTRACT

The specific impetus for this work was a conceptual design of a submarine using the toroid as the pressure hull. This work is a continuation of the work started by Bowen (1987). As such, it is hoped that a better understanding of the behavior of a toroid under hydrostatic pressure can be realized.

This work began with a review of efforts to solve complete toroidal structures. A specific toroid was then modeled in the BOSOR4 computer program to obtain displacements of the meridian under hydrostatic pressure. Functions were then derived that described the general form of these displacements. Using these functions as the assumed solution for the energy method an energy balance was made and a program was written to solve for the displacements of a generic toroid under hydrostatic loading.

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Many individuals helped me see my way through this year long study. In particular: I owe a great deal to my wife, Fran, for encouraging me to continue with the work even when things seemed hopelessly impossible and for putting up with my grouchiness when it appeared that nothing would ever come out right; to my daughter, Kristin, for understanding that her father had to work on this project instead of playing games with her; and finally to my thesis advisor, Professor Karr, who took the time to listen to some of my harebrained ideas and point out where I had deviated from normal theory and who gave me the final push in the right direction that enabled me to come up with useful results.

TABLE OF CONTENTS

4

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

The purpose of this work was to examine the response of a thin circular toroidal shell under external hydrostatic pressure. The approach used was to model a specific toroid in the BOSOR4 computer program (Bushnell, 1977) and then use the output results for displacements to generate functions that modeled these displacements. Using the functions obtained that model the displacements of the toroid under hydrostatic pressure, an analytical analysis was performed with these functions used as the assumed solution for the energy method. The energy method generated the equations necessary to solve for the displacements of a generic circular toroid. A program was then written to solve the equations generated in the energy method so that displacements for any specific toroid under hydrostatic pressure could be obtained.

The analysis conducted only considers displacements in the linear elastic range. For the shell being analyzed the following assumptions were made:

1. The material of the shell is isotropic and homogeneous.

2. The thickness of the shell is constant.

3. The thickness of the shell is small compared to the radii of curvature.

4. The displacements are symmetric about the X-Y plane (see Figure 1.1).

5. Thin shell theory holds: normals to the undeformed surface

6

remain normal.

For a symmetrically loaded shell, with the above assumption of symmetric deformation, a small displacement of any point can be resolved into two components: v in the direction of the tangent to the meridian and u in the direction of the normal to the middle surface (Timoshenko and Woinowsky-Krieger, 1959). Therefore, the only displacements discussed throughout the text will be u and v.

The toroid's geometry is straight forward but in order to be consistent throughout the text the following definitions will be made and are illustrated in Figure 1.1:

DEFINITIONS:

R : Major radius of toroid. Radius of rotation about the Z axis of the circle of radius r to form the toroid.

r : Minor radius of toroid. Radius of the circle which is rotated about the Z axis.

h : Thickness of the shell (assumed to be constant).

θ : Angle of rotation measured counter clockwise from the X-Y plane at a distance R + r from the origin.

ϕ : Angle of rotation about the Z axis measured counter clockwise from the positive X axis.

Figure 1.1

Description of Geometry
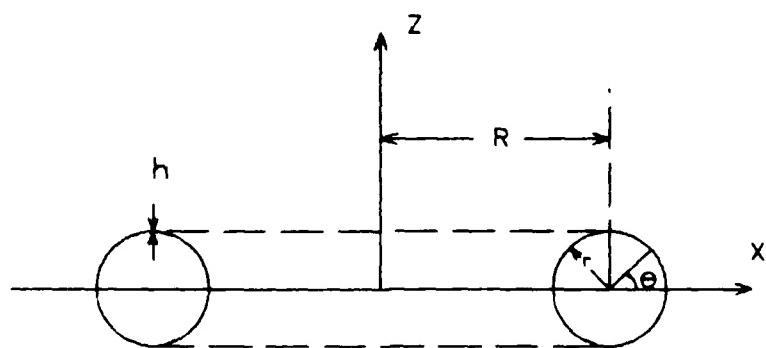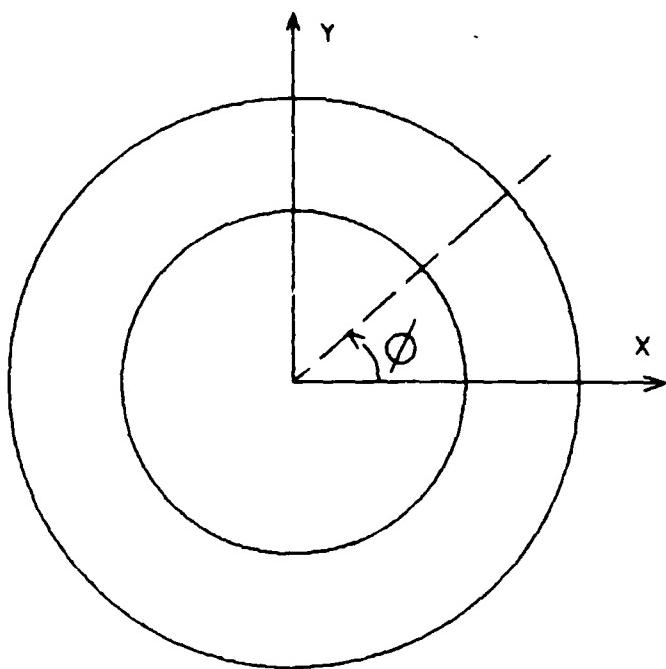
CHAPTER 2
COMPUTER MODELING

A computer solution for the displacements of the toroid under external hydrostatic pressure was sought to determine the shape of these displacements for a specific toroid. The computer program used to model the toroid was BOSOR4 (Bushnell, 1977). The BOSOR4 computer program is a hybrid finite difference-finite element program used to analyze complex shells of revolution. This program was developed by David Bushnell at Lockheed Missiles & Space Co., Palo Alto, California.

Although specific reference was not found that indicated that BOSOR4 could adequately model a toroid, it was believed that since BOSOR5, a similar program that considers elastic-plastic material behavior, had been used to model a toroid that BOSOR4 would also adequately model this structure (Bushnell, 1985).

The specific toroid that was analyzed with BOSOR4 was the toroid specified as the default values by Bowen (1987). The toroid geometry used was as follows:

R = 8 inches

r = 2 inches

h = 0.02 inches

Young's Modulus, $E = 30 \times 10^6$ psi

Poisson's Ratio, $\upsilon = 0.3$

Modeling the toroid in BOSOR4 was accomplished by taking advantage of the symmetry of loading of the structure about the X-Y plane. Therefore, only half of the structure was discretized in the program.

9

Because only half of the structure was modeled in the program, boundary conditions had to be established at the symmetry points. The boundary conditions imposed on the structure are as follows:

@ θ = 0 radians

Tangential Displacement, (v) = 0.0

Rotation in the θ direction, $\left[\frac{\partial u}{\partial \theta}\right]$ = 0.0

@ θ = Π radians

Tangential Displacement, (v) = 0.0

Rotation in the θ direction, $\left[\frac{\partial u}{\partial \theta}\right]$ = 0.0

The normal displacement (u) is defined as positive in the direction of the outward pointing normal. The tangential displacement is defined as positive in the direction of increasing angle θ (See Figure 2.1).

Having the toroid modeled in BOSOR4, the program was run to determine the pressure required to buckle the structure. The thought being that if the buckling pressure predicted by BOSOR4 was consistent with the buckling pressures predicted by theory then the pre-buckling displacements generated by BOSOR4 could be used to predict the general form of the displacements of a generic toroid.

The buckling pressure obtained from BOSOR4 for the toroid modeled differed by less than 10% from the predicted buckling pressure as presented by Sobel and Flügge (1967). Since the buckling pressure obtained was consistent with the predicted pressure and the predicted buckling pressure was substantiated experimentally by Almroth, Sobel and Hunter (1969), it was concluded that the results from BOSOR4 for the

10

Figure 2.1

BOSOR4 Modeling

Z

V

u

r

θ

$\theta = \pi$ $\theta = 0$

X
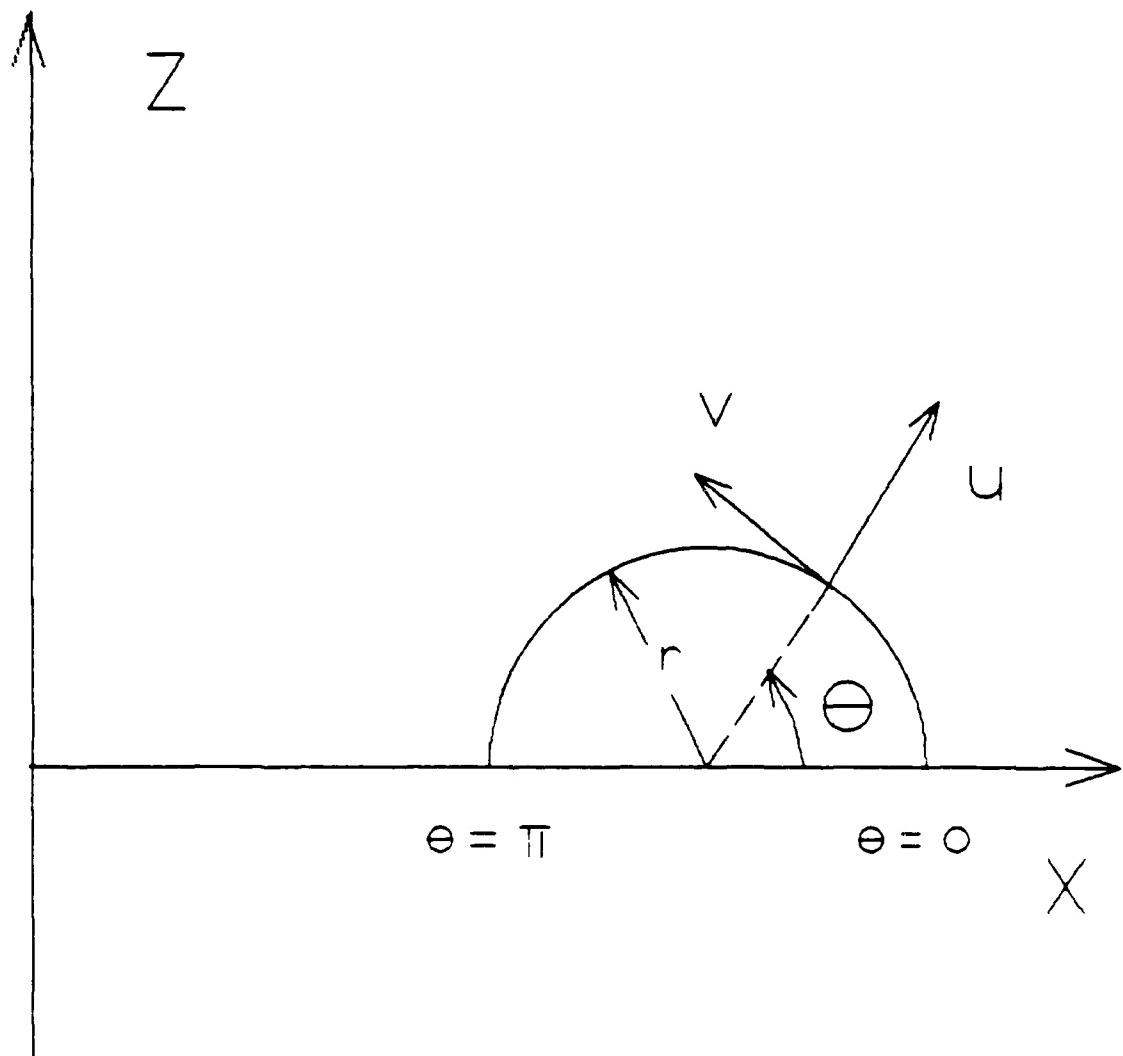
pre-buckling displacements could be used to predict the general form of the displacements of a generic toroid loaded under uniform hydrostatic pressure.

In any finite difference or finite element program an investigation into the number and spacing of the nodes used to describe the structure must be undertaken to ensure that the structure is adequately described. For this investigation, equal spacing of the nodes was used throughout.

To investigate the effect of node distribution, the number of nodes were doubled and thus the spacing was reduced by a factor of two. The results for both the magnitude of the buckling pressure and the shape and magnitude of the displacements did not change with the increase in the number of nodes. It was therefore concluded that the structure was adequately described and that the results obtained were valid.

Figure 2.2 is a graphical presentation of the normal displacement (u) and figure 2.3 is a graphical presentation of the tangential displacement (v) obtained from the BOSOR4 output for the toroid modeled.

Figure 2.2

Normal Displacement (u)

P = 1/4 Buckling Pressure

13
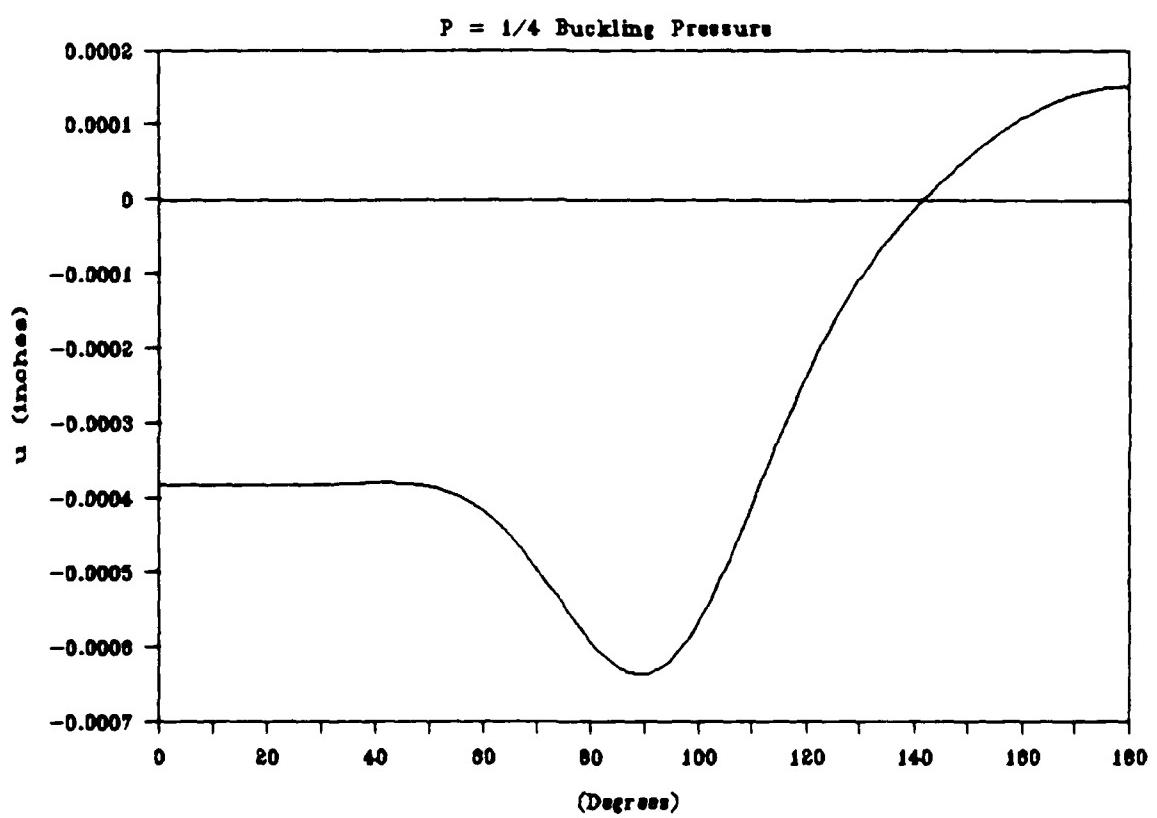
Figure 2.3
Tangential Displacement (v)

P = 1/4 Buckling Pressure

14
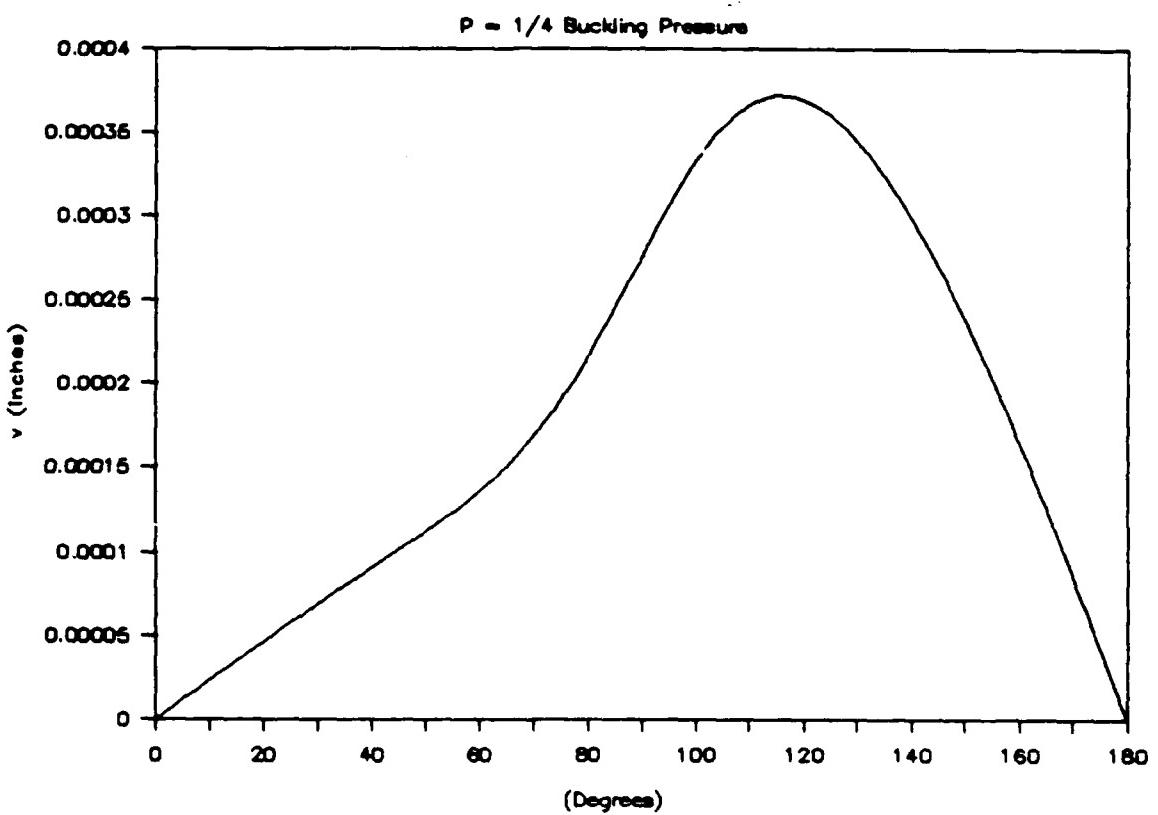
CHAPTER 3

CURVE FITTING

The next step in this study was to generate a function that would analytically describe the normal and tangential displacements of the toroid under hydrostatic loading.

First a check was made, using the BOSOR4 computer program, to ensure the displacements were linearly dependent on the applied pressure, as should be the case since this study was only considering behavior of the toroid in the linear elastic region. As can be seen from figure 3.1 the normal displacement is indeed linearly dependent on the applied pressure. Although not included here, the tangential displacement is also linearly dependent on the applied pressure.

To generate a function that would adequately describe the displacements, a form of the displacements must be assumed and then a curve fitting technique must be used to fit the data to this assumed form.

Since the displacements are considered to be symmetric about the X-Y plane and considering the way in which the positive direction for the normal displacement (u) and the tangential displacement (v) were defined in Chapter 2, the following statements can be made:

For the normal displacement (u),

$$u(\theta) - u(-\theta) \qquad (3.1)$$

which is the definition of an odd function.

For the tangential displacement (v),

$$v(\theta) - -v(-\theta) \qquad (3.2)$$

15

Figure 3.1

Incremental Loading

Normal Displacement

0.0002

0.0001

0

-0.0001   P = 1/80 B.P.

-0.0002   P = 1/40 B.P.

          P = 3/20 B.P.

-0.0003   P = 1/8 B.P.

-0.0004

-0.0005

-0.0006

u (inches)

0   20   40   60   80   100   120   140   160   180

(DEGREES)
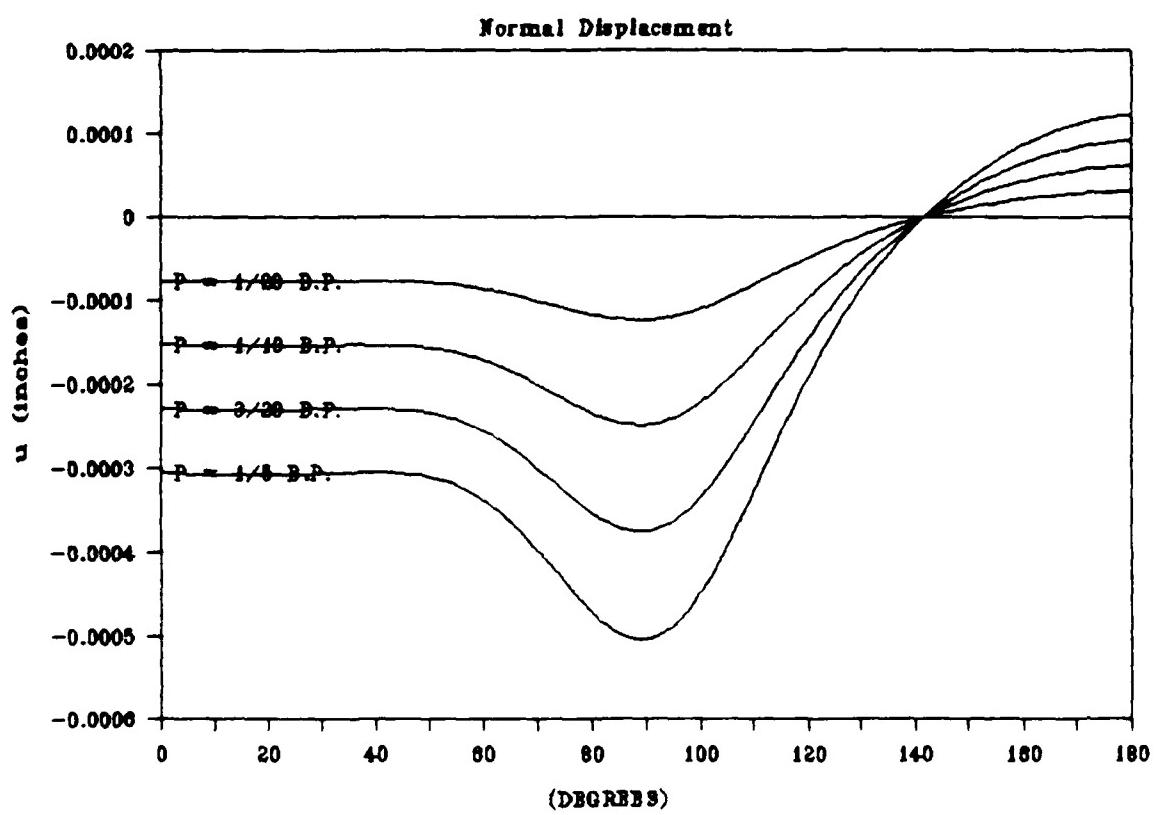
which is the definition of an even function.

The functions assumed to describe the displacements were Fourier Series for both the normal (u) and tangential (v) displacements. Since both the normal and tangential displacements have a period of $2\pi r$ (the arc length of the circular cross section) and noting the above statements about even and odd functions, the Fourier Series for the displacements can be written as follows:

$$u(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left[\frac{n\pi x}{L}\right] \qquad (3.3),$$

$$v(\theta) = \sum_{n=1}^{\infty} b_n \sin\left[\frac{n\pi x}{L}\right] \qquad (3.4).$$

For the displacements to be symmetric about the X-Y plane, or a half period of .e function, x and L can be defined as follows:

$$x = r\theta \quad ,$$

$$L = \pi r \quad .$$

From these definitions, u and v can be rewritten as:

$$u(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) \qquad (3.5),$$

$$v(\theta) = \sum_{n=1}^{\infty} b_n \sin(n\theta) \qquad (3.6).$$

While the Fourier series described above will give an exact representation of the displacements, the goal of a curve fitting routine should be to determine how many terms of the series must be used to

adequately describe the function. To determine the number of terms required to describe the displacements a least squares curve fitting program was written to be used in conjunction with the output data from BOSOR4. The programs were adapted from a curve fitting routine presented by James, Smith and Wolford (1977) and are included in Appendix A.

Using these programs to determine the coefficients of the Fourier Series it was found that five terms were inadequate to describe the displacements, as can be seen graphically in Figure 3.2, and it was determined that ten terms were required to adequately describe the normal displacements, (See Figure 3.3), and nine terms were required to describe the tangential displacements. Although only the normal displacement (u) is presented in Figures 3.2 and 3.3 similar results were obtained for the tangential displacement (v).

With the number of terms determined that are required in the Fourier series u and v can be written in final form as:

$$u(\theta) = \frac{a_0}{2} + \sum_{n=1}^{9} a_n \cos(n\theta) \qquad (3.7),$$

$$v(\theta) = \sum_{n=1}^{9} b_n \sin(n\theta) \qquad (3.8).$$

These definitions of u and v will be used in the next section as the assumed forms of the displacements of the toroid.

Figure 3.2

Five Factors

1/4 PREDICTED BUCKLING LOAD

NORMAL DISPLACEMENT

BOSOR4

CURVE FIT

(DEGREES)

0   20   40   60   80   100   120   140   160   180

Figure 3.3

Ten Factors

1/4 PREDICTED BUCKLING LOAD

NORMAL DISPLACEMENT

BOSOR4

CURVE FIT

0   20   40   60   80   100   120   140   160   180
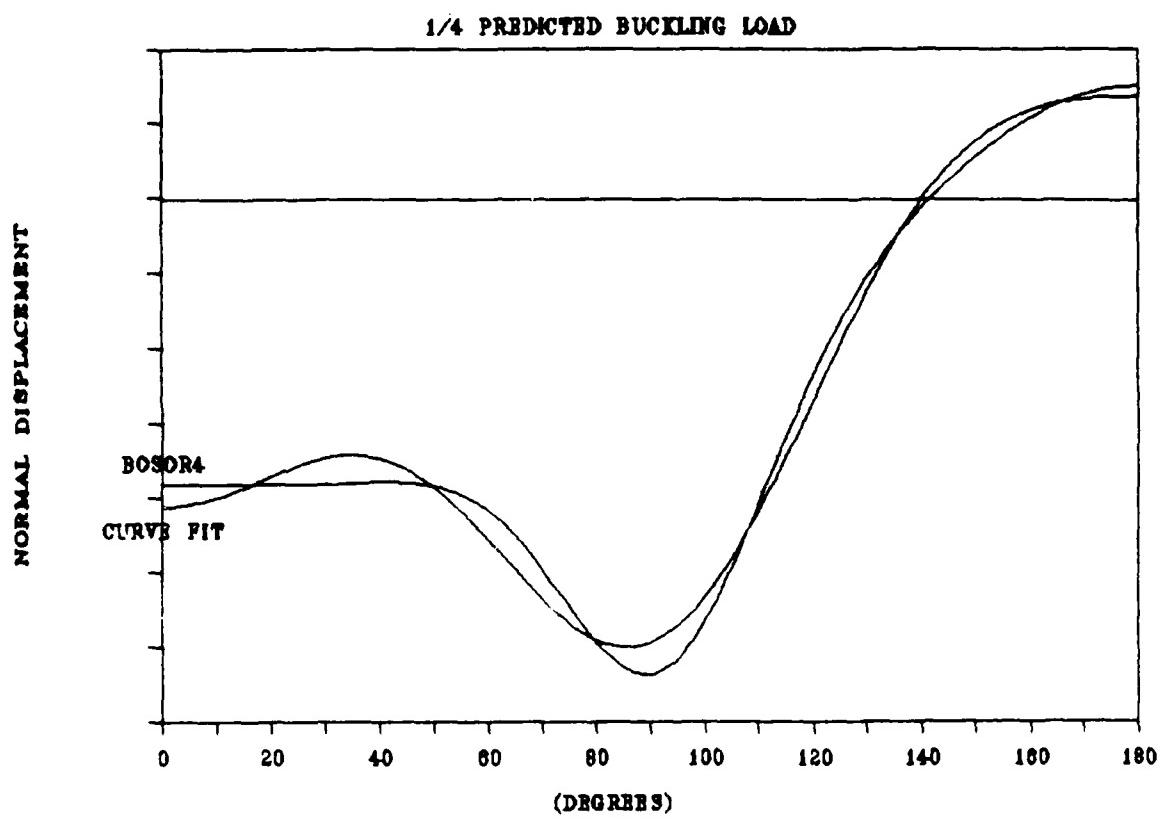
(DEGREES)
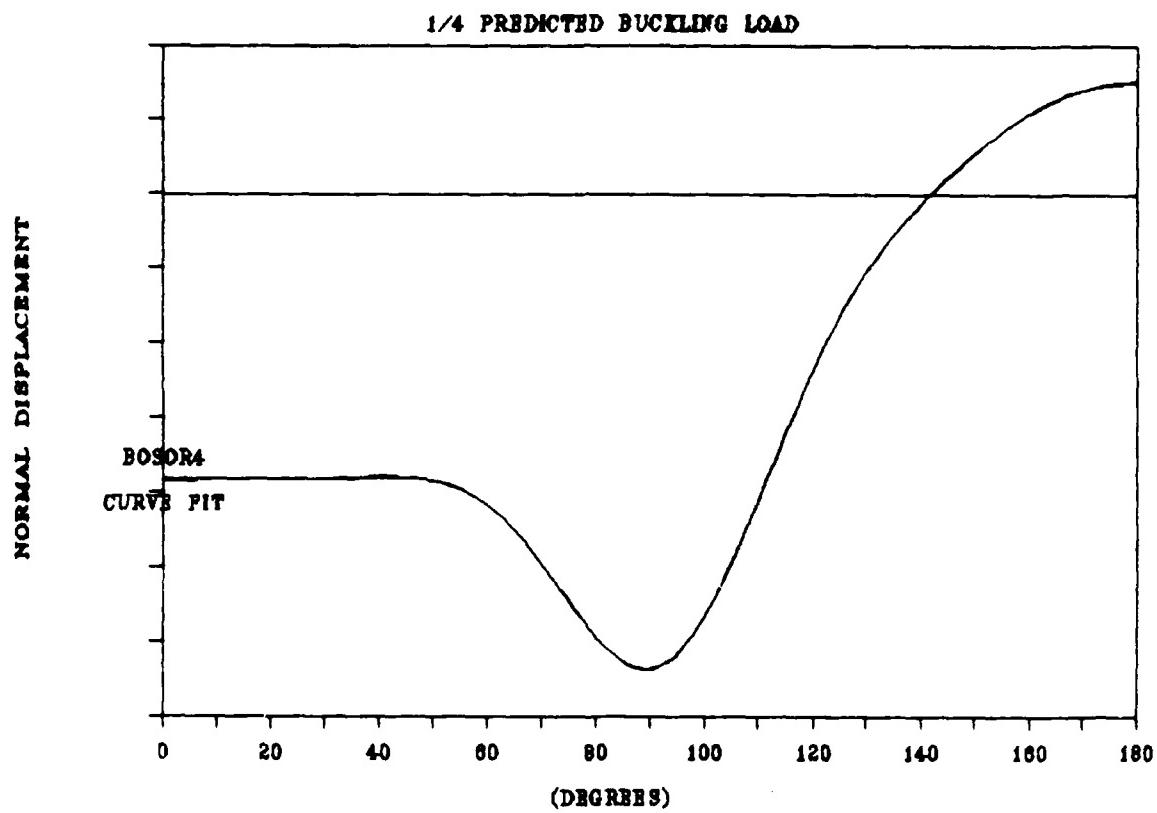
CHAPTER 4

ENERGY METHOD

4.1 Introduction:

To determine the displacements of a structure when it is subjected to an external load, the external load must be balanced by the internal forces of the structure caused by the external load.

For this study, the method of minimizing the total potential energy, as described by Shames and Dym (1985), was used. The energy method was cho..en in hopes of avoiding the singularities in the solution at $\theta = \pm \frac{\pi}{2}$,(Reissner, 1963 and Senjanovic', 1972). The total potential energy is defined as:

$$\Pi = U - W \qquad (4.1.1)$$

where U and W are defined as;

U : internal energy of the structure,

W : potential energy of the applied loads.

To determine the minimum potential energy, and thereby determine the displacements caused by the external load, the first variation of the total potential energy is set to zero.

$$\delta^{(1)}\Pi = 0 \qquad (4.1.2)$$

The first variation of the total potential energy can be defined as the following partial differential equation:

$$\frac{\partial \Pi}{\partial x_i} = 0 \qquad (4.1.3)$$

where x_i is any general parameter used to describe U and W.

Since the first variation of the total potential energy is set to zero,

21

the equation to be solved to determine the displacements of the structure can be defined as;

$$\frac{\partial W}{\partial x_i} - \frac{\partial U}{\partial x_i} \qquad (4.1.4).$$

In order to solve the above equation for the displacements caused by the applied load, U and W must be defined as functions of the displacements.

As described by Bushnell (1984), the strain energy of the structure can be defined as follows:

$$U - U_s + U_c \qquad (4.1.5)$$

where;

   U_s : strain energy due to elongation and changes in curvature,

   U_c : strain energy due to constraints or boundary conditions.

For the toroid, the symmetry about the X-Y Plane will be taken advantage of and only half of the structure will be analyzed. The following additional geometric definitions are required for the analysis (See Figure 4.1).

DEFINITIONS:

   a : Ratio of the major and the minor radius of the toroid. $a - \frac{R}{r}$

   r_T : Distance from the Z axis to the meridian of the toroid.

   $r_T - R + r \cos(\theta) - r \left[a + \cos(\theta) \right]$

   r_2 : Radius of curvature in the θ direction.

   $r_2 - \dfrac{r_T}{\cos(\theta)} - \dfrac{R + r \cos(\theta)}{\cos(\theta)}$

22

Figure 4.1

Additional Geometry
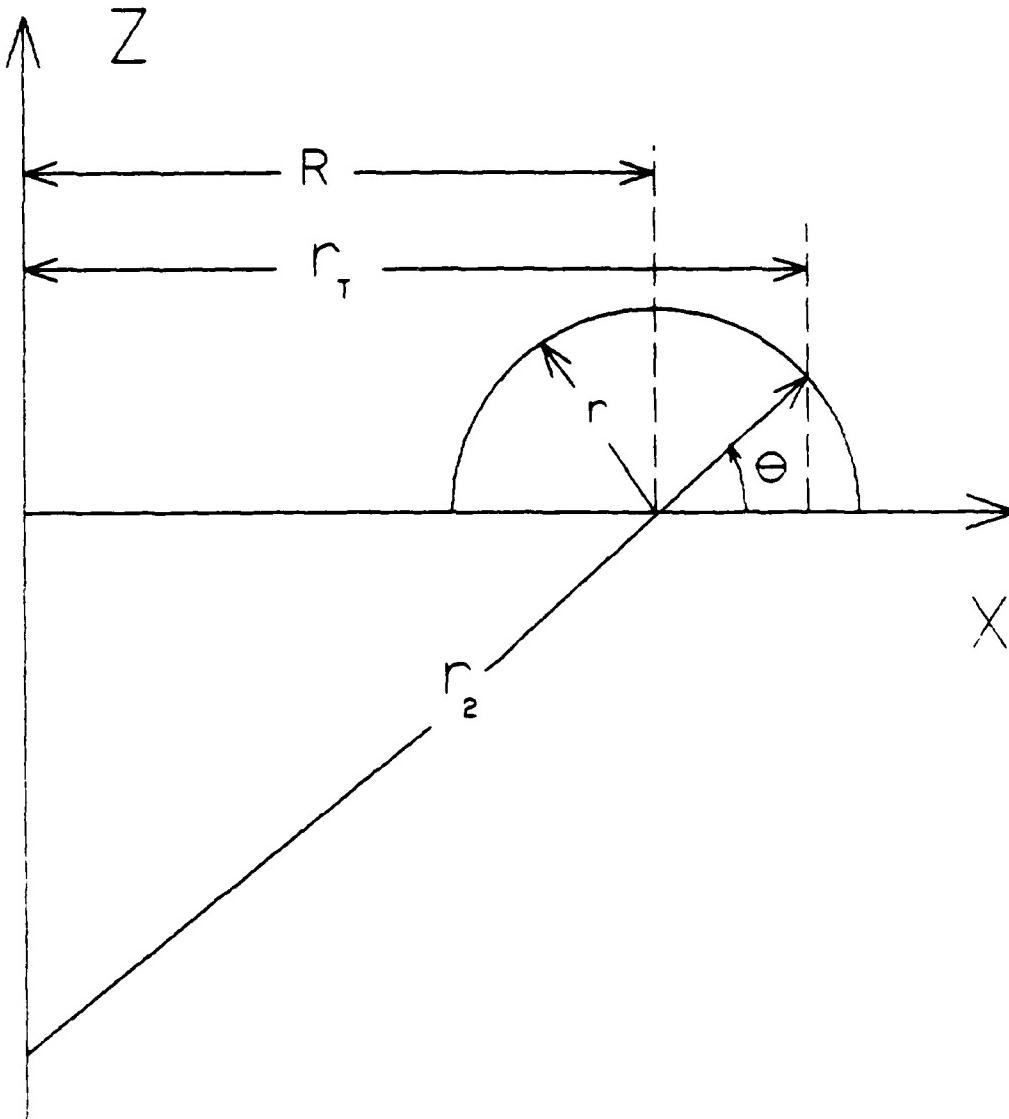
4.2 Strain Energy:

The strain energy due to elongation and changes in curvature can be defined as:

$$U_s = \frac{1}{2} \int_v \sigma_{ij} \epsilon_{ij} \, dv \quad , \quad i,j = \theta \, , \, \phi \qquad (4.2.1).$$

The strain energy due to constraints, U_c , will be addressed later.

Because this study is only considering hydrostatic loading of the toroid, the loading will always be normal to the surface of the toroid. Therefore, there will not be any shear stresses or shear resultants from the loading imposed.

$$\sigma_{ij} = \epsilon_{ij} = 0 \, , \text{ if } i \neq j.$$

The θ and ϕ directions are thus principle directions and the strains can be represented by reference surface strains and changes in curvature. The strain energy can be defined in terms of forces and moments which are defined as an integral of the stresses through the thickness of the shell. The strain energy can then be represented by a surface integral as follows:

$$U_s = \frac{1}{2} \int_s \left(N_{ij} \epsilon_{ij} + M_{ij} \kappa_{ij} \right) ds \qquad (4.2.2).$$

For the case of hydrostatic loading the strain energy becomes:

$$U_s = \frac{1}{2} \int_s \left[\left(N_\theta \epsilon_\theta + N_\phi \epsilon_\phi \right) + \left(M_\theta \kappa_\theta + M_\phi \kappa_\phi \right) \right] ds \qquad (4.2.3)$$

where the forces and moments are defined as follows (Timoshenko and Woinowsky - Krieger, 1959):

$$N_\theta = \int_t \sigma_\theta \left[1 - \frac{z}{r} \right] dz \qquad (4.2.4),$$

$$N_\phi = \int_t \sigma_\phi \left[1 - \frac{z}{r_2} \right] dz \qquad (4.2.5),$$

24

$$M_\theta = \int_t \sigma_\theta \, z \left(1 - \frac{z}{r} \right) dz \qquad (4.2.6),$$

$$M_\phi = \int_t \sigma_\phi \, z \left(1 - \frac{z}{r_2} \right) dz \qquad (4.2.7).$$

Since the assumption has been made that normals to the undeformed middle surface remain normal, the strains as a function of thickness can be expressed in terms of reference surface strains and changes in curvature (Timoshenko and Woinowsky-Krieger, 1959).

$$\epsilon_{\theta_t} = \epsilon_\theta - z \, \kappa_\theta \qquad (4.2.8)$$

$$\epsilon_{\phi_t} = \epsilon_\phi - z \, \kappa_\phi \qquad (4.2.9)$$

The terms, ϵ_{θ_t} and ϵ_{ϕ_t}, represent the total strain in the θ and ϕ directions and κ_θ and κ_ϕ represent the changes in curvature of the reference surface.

From Hook's Law the stresses can be defined as follows:

$$\sigma_\theta = \frac{E}{(1-v^2)} \left[\epsilon_{\theta_t} + v\epsilon_{\phi_t} \right] \qquad (4.2.10)$$

$$\sigma_\phi = \frac{E}{(1-v^2)} \left[\epsilon_{\phi_t} + v\epsilon_{\theta_t} \right] \qquad (4.2.11)$$

substituting the expressions for ϵ_{θ_t} and ϵ_{ϕ_t},

$$\sigma_\theta = \frac{E}{(1-v^2)} \left[\epsilon_\theta + v\epsilon_\phi - z\left(\kappa_\theta + v\kappa_\phi \right) \right] \qquad (4.2.12),$$

$$\sigma_\phi = \frac{E}{(1-v^2)} \left[\epsilon_\phi + v\epsilon_\theta - z\left(\kappa_\phi + v\kappa_\theta \right) \right] \qquad (4.2.13).$$

Taking the middle surface as the reference surface and neglecting the terms, $\frac{z}{r}$ and $\frac{z}{r_2}$, as small as compared to unity, integration over

25

the thickness of the shell can be represented by:

$$N_\theta = \int_{-h/2}^{+h/2} \frac{E}{(1-\upsilon^2)} \left[\epsilon_\theta + \upsilon\epsilon_\phi - z\left(\kappa_\theta + \nu\kappa_\phi \right) \right] dz \qquad (4.2.14),$$

$$N_\theta = \frac{Eh}{(1-\upsilon^2)} \left[\epsilon_\theta + \upsilon\epsilon_\phi \right] \qquad (4.2.15),$$

$$N_\phi = \int_{-h/2}^{+h/2} \frac{E}{(1-\upsilon^2)} \left[\epsilon_\phi + \upsilon\epsilon_\theta - z\left(\kappa_\phi + \nu\kappa_\theta \right) \right] dz \qquad (4.2.16),$$

$$N_\phi = \frac{Eh}{(1-\upsilon^2)} \left[\epsilon_\phi + \upsilon\epsilon_\theta \right] \qquad (4.2.17),$$

$$M_\theta = \int_{-h/2}^{+h/2} \frac{E}{(1-\upsilon^2)} \left[\epsilon_\theta + \upsilon\epsilon_\phi - z\left(\kappa_\theta + \nu\kappa_\phi \right) \right] z\, dz \qquad (4.2.18),$$

$$M_\theta = \frac{-Eh^3}{12(1-\upsilon^2)} \left[\kappa_\theta + \nu\kappa_\phi \right] \qquad (4.2.19),$$

$$M_\phi = \int_{-h/2}^{+h/2} \frac{E}{(1-\upsilon^2)} \left[\epsilon_\phi + \upsilon\epsilon_\theta - z\left(\kappa_\phi + \nu\kappa_\theta \right) \right] z\, dz \qquad (4.2.20),$$

$$M_\phi = \frac{-Eh^3}{12(1-\upsilon^2)} \left[\kappa_\phi + \nu\kappa_\theta \right] \qquad (4.2.21).$$

Combining the expressions for the forces and the moments and substituting these back into the expression for the strain energy, the strain energy becomes:

$$U_s = \frac{1}{2} \int_s \left[\frac{Eh}{(1-\upsilon^2)} \left(\epsilon_\theta^2 + 2\upsilon\epsilon_\theta\epsilon_\phi + \epsilon_\phi^2 \right) - \right.$$

$$\left. \frac{Eh^3}{12(1-\upsilon^2)} \left(\kappa_\theta^2 + 2\upsilon\kappa_\theta\kappa_\phi + \kappa_\phi^2 \right) \right] ds \qquad (4.2.22).$$

4.2.1 Strains:

The next step is to define the surface strains in terms of the displacements u and v. This will be accomplished by following the

26

presentation by Timoshenko and Woinowsky-Krieger,(1959).

Considering an element AB of the meridian, (See Figure 4.2), the strain in the θ direction can be defined as follows:

ϵ_θ due to v,

$$\epsilon = \frac{v + \frac{\partial v}{\partial \theta} d\theta - v}{r \, d\theta} = \frac{1}{r} \frac{\partial v}{\partial \theta} \qquad (4.2.1.1)$$

ϵ_θ due to u can be represented by the average change in length of the element divided by the original length,

$$\epsilon = \frac{\left(u + \frac{1}{2} \frac{\partial u}{\partial \theta} d\theta \right) d\theta}{r \, d\theta} \qquad (4.2.1.2).$$

The component, $\frac{1}{2} \frac{\partial u}{\partial \theta} d\theta$, will be neglected as a small quantity of higher order. The total strain in the θ direction can then be represented as:

$$\epsilon_\theta = \frac{1}{r} \left(u + \frac{\partial v}{\partial \theta} \right) \qquad (4.2.1.3).$$

The strain in the ϕ direction can be defined as the change in r_T due to u and v divided by the original length of the element (See Figure 4.3).

$$\Delta r_T = \left(u \cos(\theta) - v \sin(\theta) \right) d\phi , \text{ and}$$

$$\epsilon_\phi = \frac{\left(u \cos(\theta) - v \sin(\theta) \right) d\phi}{r_T \, d\phi} \qquad (4.2.1.4),$$

$$\epsilon_\phi = \frac{1}{r (a + \cos(\theta))} \left[u \cos(\theta) - v \sin(\theta) \right] \qquad (4.2.1.5).$$

4.2.2 Change in Curvature:

The change in curvature of the middle surface will now be defined.

Again considering the element AB of the meridian, (See Figure 4.2), the rotation in the θ direction can be defined as follows:

Figure 4.2

Differential Element

28
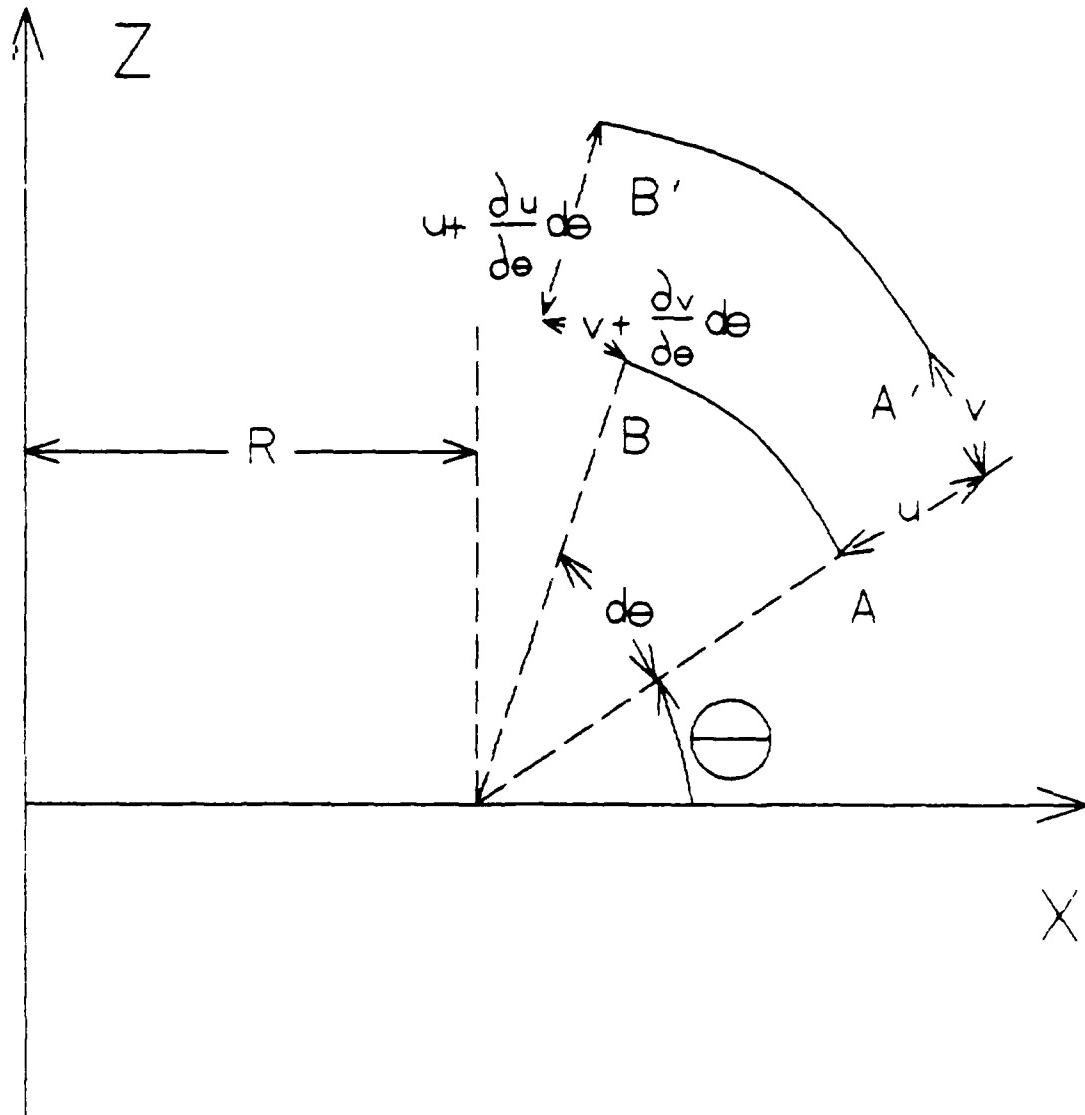
Figure 4.3

Radial Displacement
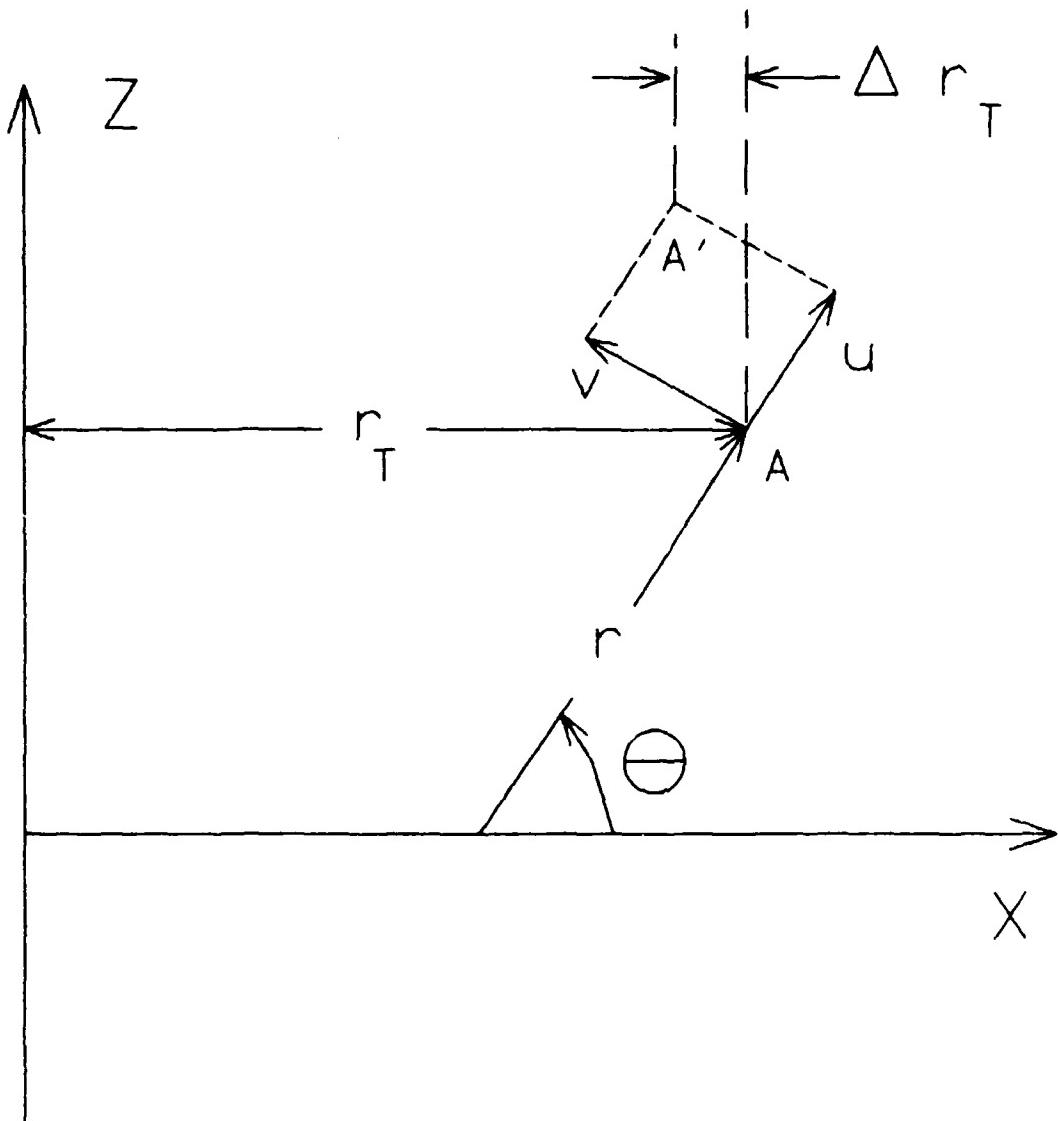
β due to v,

$$\beta = \frac{v}{r} \, d\theta \qquad\qquad (4.2.2.1)$$

β due to u,

$$\beta = \frac{-\frac{\partial u}{\partial \theta} \, d\theta}{r} = \frac{-1}{r} \, \frac{\partial u}{\partial \theta} \, d\theta \qquad\qquad (4.2.2.2).$$

The total rotation in the θ direction is;

$$\beta_\theta = \frac{1}{r} \left[v - \frac{\partial u}{\partial \theta} \right] \, d\theta \qquad\qquad (4.2.2.3).$$

The change in curvature in the θ direction can be represented by the change in rotation in the θ direction divided by the undeformed arc length in the θ direction.

$$\kappa_\theta = \frac{1}{r} \, \frac{\partial \beta_\theta}{\partial \theta} = \frac{1}{r^2} \left(\frac{\partial v}{\partial \theta} - \frac{\partial^2 u}{\partial \theta^2} \right) \qquad\qquad (4.2.2.4)$$

Because the assumption has been made that the deformations will be symmetric, the rotation in the ϕ direction will be opposite to the rotation in the θ direction in magnitude and rotated into the ϕ surface by sin (θ).

$$\beta_\phi = - \beta_\theta \, \sin \, (\theta) = \frac{1}{r} \left[\frac{\partial u}{\partial \theta} - v \right] \, \sin \, (\theta) \, d\phi \qquad\qquad (4.2.2.5)$$

Although the rotation in the ϕ direction is constant in ϕ , it does vary in the θ direction.   Therefore, the change in curvature due to loading can be represented by the rotation in the ϕ direction divided by the undeformed arc length in the ϕ direction.

$$\kappa_\phi = \frac{\beta_\phi}{r \, (a + \cos(\theta)) \, d\phi} \qquad\qquad (4.2.2.6)$$

30

and substituting the expression for β_ϕ ,

$$\kappa_\phi = \frac{\sin(\theta)}{r^2(a + \cos(\theta))} \left[\frac{\partial u}{\partial \theta} - v \right] \qquad (4.2.2.7).$$

4.2.3 Constraint Energy:

A complete toroid has no constraint or boundary conditions when loaded only by hydrostatic pressure. However, since only half of the toroid is being analyzed here, boundary conditions must be imposed at the symmetry points, $\theta = 0$ and $\theta = \pi$. The boundary conditions that are required at these two points are:

The vertical displacement is zero, $v = 0$,

and the rotation in the θ direction is zero, $\frac{\partial u}{\partial \theta} = 0$.

As Bushnell (1984) points out the constraint energy can be accounted for with the introduction of Lagrange multipliers and can be represented as follows:

$$U_c = \lambda_{0u} \frac{\partial u}{\partial \theta}(0) + \lambda_{0v} v(0) + \lambda_{\pi u} \frac{\partial u}{\partial \theta}(\pi) + \lambda_{\pi v} v(\pi) \qquad (4.2.3.1).$$

In this formulation of the constraint energy the Lagrange multipliers are unknowns and are solved for along with the dis$_\iota$lacements when the total potential energy is minimized. The actual form of the constraint energy will be discussed later.

4.3 Potential Energy of Applied Loads:

The potential energy of the applied loads is simply the work done by these loads, which can be represented by the applied load multiplied

31

by the deflection in the direction of the applied load. For the case of hydrostatic loading, the applied load is always normal to the surface and the displacement in the direction of this load is -u for this study. Therefore, the potential energy of the applied hydrostatic load can be represented as:

$$W = - \int_s P \, u \, ds \qquad (4.3.1)$$

where P is the external hydrostatic pressure applied and is positive in the direction of -u.

4.4 Minimization of the Total Potential Energy:

From section 4.1 it was shown that to minimize the total potential energy the following equation needed to be solved;

$$\frac{\partial \Pi}{\partial x_i} = 0 \qquad (4.4.1),$$

and it was shown that the above equation could be represented by the following equation:

$$\frac{\partial W}{\partial x_i} = \frac{\partial U}{\partial x_i} \qquad (4.4.2).$$

To present the form of the above equations the following simplifications in notation will be used:

$$u' = \frac{\partial u}{\partial \theta} \, , \qquad u'' = \frac{\partial^2 u}{\partial \theta^2} \, , \qquad v' = \frac{\partial v}{\partial \theta} \, ,$$

$$C = \frac{Eh}{(1 - v^2)} \, , \qquad D = \frac{Eh^3}{12(1 - v^2)} \, .$$

32

4.4.1 Internal Energy:

Since it has been shown that none of the terms in the internal energy depend on ϕ the surface integral can be rewritten and integrated with respect to ϕ.

$$U_s = \frac{1}{2} \int_0^\pi \int_0^{2\pi} \left[\frac{Eh}{(1-\upsilon^2)} \left(\epsilon_\theta^2 + 2\upsilon\epsilon_\theta\epsilon_\phi + \epsilon_\phi^2 \right) - \right.$$

$$\left. \frac{Eh^3}{12(1-\upsilon^2)} \left(\kappa_\theta^2 + 2\upsilon\kappa_\theta\kappa_\phi + \kappa_\phi^2 \right) \right] r\Big(a + \cos(\theta)\Big) d\phi \ rd\theta \qquad (4.4.1.1)$$

Which when integrated with respect to ϕ becomes,

$$U_s = \pi \ r^2 \int_0^\pi \left[\frac{Eh}{(1-\upsilon^2)} \left(\epsilon_\theta^2 + 2\upsilon\epsilon_\theta\epsilon_\phi + \epsilon_\phi^2 \right) - \right.$$

$$\left. \frac{Eh^3}{12(1-\upsilon^2)} \left(\kappa_\theta^2 + 2\upsilon\kappa_\theta\kappa_\phi + \kappa_\phi^2 \right) \right] \Big(a + \cos(\theta)\Big) d\theta \qquad (4.4.1.2).$$

Using the above simplifications in notation and substituting the expressions presented in section 4.2.1 and 4.2.2 the strain energy becomes:

$$U_s = \pi \ r^2 \int_0^\pi \left[\frac{C}{r^2} \left[\left(u^2 + 2uv' + v'^2 \right) + \right. \right.$$

$$\frac{2\upsilon}{(a + \cos(\theta))} \left((u^2+uv')\cos(\theta) - (uv + vv')\sin(\theta) \right)$$

$$+ \frac{1}{(a + \cos(\theta))^2} \left(u^2\cos^2(\theta) - 2uv \cos(\theta)\sin(\theta) + \right.$$

$$\left. \left. v^2\sin^2(\theta) \right) \right] - \frac{D}{r^4} \left[\left(v'^2 - 2v'u'' + u''^2 \right) + \right.$$

$$\frac{2 \ \upsilon \ \sin(\theta)}{(a + \cos(\theta))} \left(v'u' - vv' - u'u'' + vu'' \right) +$$

33

$$\frac{\sin^2(\theta)}{(a + \cos(\theta))^2} 2\left(u'^2 - 2u'v + v^2\right)\Big]\Big] \left(a+\cos(\theta)\right) d\theta \qquad (4.4.1.3).$$

The above equation represents the strain energy, U_s. However, for minimization of the total potential energy the expression that is needed is $\frac{\partial U}{\partial x_i}$.

$$\frac{\partial U_s}{\partial x_i} = \pi \int_0^\pi \Bigg[C \Bigg[\left(2u\frac{\partial u}{\partial x_i} + 2(u\frac{\partial v'}{\partial x_i} + v'\frac{\partial u}{\partial x_i}) + 2v'\frac{\partial v'}{\partial x_i} \right) +$$

$$\frac{2\nu}{(a + \cos(\theta))}\Big((2u\frac{\partial u}{\partial x_i} + u\frac{\partial v'}{\partial x_i} + v'\frac{\partial u}{\partial x_i})\cos(\theta) - (u\frac{\partial v}{\partial x_i} + v\frac{\partial u}{\partial x_i} + v\frac{\partial v'}{\partial x_i} + v'\frac{\partial v}{\partial x_i})\sin(\theta)\Big)$$

$$+\frac{1}{(a + \cos(\theta))^2}\Big(2u\frac{\partial u}{\partial x_i}\cos^2(\theta) - 2(u\frac{\partial v}{\partial x_i} + v\frac{\partial v}{\partial x_i})\cos(\theta)\sin(\theta) + 2v\frac{\partial v}{\partial x_i}\sin^2(\theta) \Big) \Bigg]$$

$$- \frac{D}{r^2} \Bigg[\left(2v'\frac{\partial v'}{\partial x_i} - 2(v'\frac{\partial u''}{\partial x_i} + u''\frac{\partial v'}{\partial x_i}) + 2u''\frac{\partial u''}{\partial x_i} \right) +$$

$$\frac{2\,\nu\,\sin(\theta)}{(a + \cos(\theta))}\left(v'\frac{\partial u'}{\partial x_i} + u'\frac{\partial v'}{\partial x_i} - v\frac{\partial v'}{\partial x_i} - v'\frac{\partial v}{\partial x_i} - u'\frac{\partial u''}{\partial x_i} - u''\frac{\partial u'}{\partial x_i} + v\frac{\partial u''}{\partial x_i} + u''\frac{\partial v}{\partial x_i} \right) +$$

$$\frac{\sin^2(\theta)}{(a + \cos(\theta))^2}\Big(2u'\frac{\partial u'}{\partial x_i} - 2(u'\frac{\partial v}{\partial x_i} + v\frac{\partial u'}{\partial x_i}) + 2v\frac{\partial v}{\partial x_i} \Big) \Bigg]\Bigg] \left(a+\cos(\theta)\right) d\theta \qquad (4.4.1.4)$$

The above equation represents the first term on the right hand side of the following equation that will be needed to minimize the total potential energy and thereby solve for the displacements,

$$\frac{\partial W}{\partial x_i} = \frac{\partial U_s}{\partial x_i} + \frac{\partial U_c}{\partial x_i} \qquad (4.4.1.5).$$

4.4.2 External Energy:

Similar to the internal energy, the external energy does not depend on ϕ and the surface integral can be rewritten and integrated with respect to ϕ.

$$W = - \int_0^\pi \int_0^{2\pi} \Big[P\,u \Big] r\Big(a + \cos(\theta) \Big) \, d\phi \; r \, d\theta \qquad (4.4.2.1)$$

Which when integrated with respect to ϕ becomes,

$$W = -2P\pi r^2 \int_0^\pi u \left(a + \cos(\theta) \right) d\theta \qquad (4.4.2.2).$$

For minimization of the total potential energy the expression that is needed is $\frac{\partial W}{\partial x_i}$.

$$\frac{\partial W}{\partial x_i} = -2P\pi r^2 \int_0^\pi \frac{\partial u}{\partial x_i} \left(a + \cos(\theta) \right) d\theta \qquad (4.4.2.3)$$

This completes the expressions needed for the minimization of the total potential energy, with the exception of the constraint energy, U_c .

4.5 Assumed Functions:

In Chapter 3 it was shown that the displacements u and v could be represented by a Fourier series. These functions of θ will be used as the assumed form of the displacements and the coefficients of the Fourier series will be solved for from the equations presented in the minimization of the total potential energy. The assumed functions for u and v are different from those presented in Chapter 3 in that the constant term in the expression for u has been included inside the summation.

$$u(\theta) = \sum_{n=0}^{9} a_n \cos (n\theta) \qquad (4.5.1),$$

$$v(\theta) = \sum_{n=0}^{9} b_n \sin (n\theta) \qquad (4.5.2).$$

The above expressions for u and v involve 19 unknown coefficients that will have to be solved for to describe the displacements.

Using these expressions for u and v the constraint energy, U_c ,

35

discussed in section 4.2.3 can now be addressed. The constraints needed were the boundary conditions at the symmetry points, $\theta = 0$ and $\theta = \pi$. These boundary conditions are:

$v = 0$. Since $\sin(n\theta) = 0$, for $\theta = 0$ and $\sin(n\pi) = 0$, for all n, these boundary conditions are satisfied by the assumed form of v and additional constraints are not needed.

$\frac{\partial u}{\partial \theta} = 0$. Since $\frac{\partial u}{\partial \theta} = -\sum_{n=0}^{9} n\, a_n \sin(n\theta)$, then $\frac{\partial u}{\partial \theta} = 0$ at $\theta = 0$ and

$\theta = \pi$ for the same reasons as stated above for v.

Since no additional constraints are needed to meet the boundary conditions, Lagrange multipliers are not needed and the constraint energy, $U_c = 0$.

4.6 Solution:

With the assumed forms of the displacements defined, the problem is now restricted to solving for the 19 unknown coefficients of the Fourier series. Using the following substitutions the equations can be set up to solve for these coefficients:

$$u = \sum_{n=0}^{9} a_n \cos(n\theta) \tag{4.6.1},$$

$$u' = -\sum_{n=0}^{9} n\, a_n \sin(n\theta) \tag{4.6.2},$$

$$u'' = -\sum_{n=0}^{9} n^2\, a_n \cos(n\theta) \tag{4.6.3},$$

$$v = \sum_{n=0}^{9} b_n \sin(n\theta) \tag{4.6.4},$$

$$v' = \sum_{n=0}^{9} n\, b_n \cos(n\theta) \tag{4.6.5}.$$

36

Substituting the above expressions into the following equation will then represent the 19 equations needed to solve for the 19 unknown coefficients:

$$\frac{\partial W}{\partial x_i} = \frac{\partial U_s}{\partial x_i} \qquad (4.6.6).$$

Which can be represented in matrix notation as follows:

$$[\,D\,] = [\,F\,]\,[\,x_i\,] \qquad (4.6.7)$$

where the above terms are defined as:

D : a 19 x 1 column consisting of the terms, $\frac{\partial W}{\partial x_i}$,

F : a 19 x 19 matrix consisting of the terms, $\frac{\partial U_s}{\partial x_i}$,

x_i : a 19 x 1 column consisting of the terms, a_n and b_n. Where x_1 through x_{10} are the 10 a_n's and x_{11} through x_{19} are the 9 b_n's.

To solve for the x_i's, the coefficients of the Fourier series, the following matrix operation needs to be performed:

$$[\,x_i\,] = [\,F\,]^{-1}\,[\,D\,] \qquad (4.6.8).$$

The individual elements in F can be determined by performing the integration from 0 to π of the expression for $\frac{\partial U_s}{\partial x_i}$ as presented in section 4.4.1. Similarly , the individual elements in D can be determined by performing the integration of the expression for $\frac{\partial W}{\partial x_i}$ as presented in section 4.4.2. To perform the above integrations, as well as inverting the F matrix and solving for the x_i's, a program was written. This "Toroid Program" is listed in Appendix B.

37

The "Toroid Program" was run to compare the results with those obtained using BOSOR4. As can be seen from Figures 4.4 and 4.5, the results obtained for the displacements are comparable to those obtained from BOSOR4 and are of the same general shape. Additionally, when the internal energy of the results are compared, the "Toroid Program" is only 4.3% greater than that of the BOSOR4 solution. This difference can be accounted for by the selection of the functions chosen to represent u and v. The functions chosen for u and v only model the displacements and do not model the change in displacements or the curvature of the toroid.

4.7 Forces and Moments:

The next step was to look at the predictions for the forces, N_θ and N_ϕ, and the moments, M_θ and M_ϕ, and compare them with the results from BOSOR4.

The forces and moments predicted by the "Toroid Program" did not corollate well with the results obtained from BOSOR4. As can be seen from Figures 4.6 and 4.7, the average of the predicted forces is approximately the same as the forces obtained from BOSOR4. However, as can be seen from Figures 4.8 through 4.11, the predicted moments were several orders of magnitude less than those obtained from BOSOR4.

38

Figure 4.4

Normal Displacement (u)

1/4 PREDICTED BUCKLING LOAD

NORMAL DISPLACEMENT

TOROID PROGRAM

BOSOR4

0   20   40   60   80   100   120   140   160   180

(DEGREES)
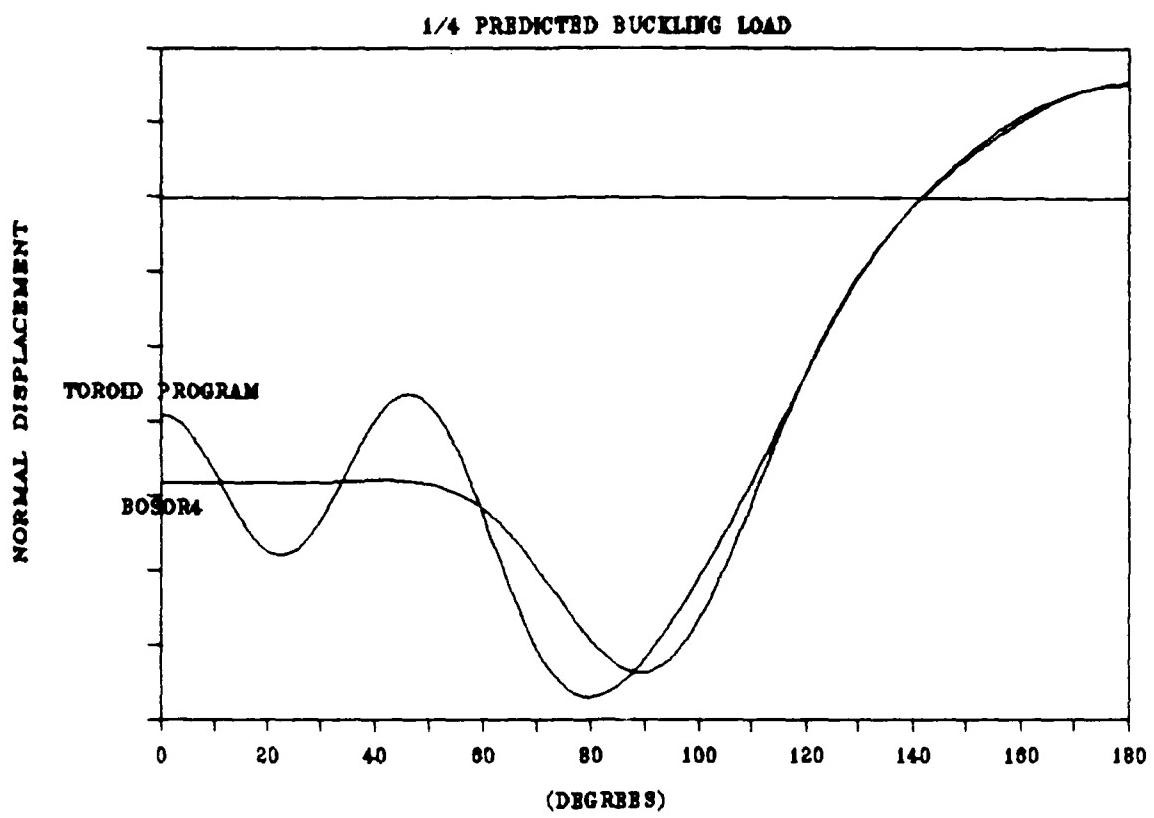
Figure 4.5

Tangential Displacement (v)

1/4 PREDICTED BUCKLING LOAD

Figure 4.6

N_{θ} vs θ

1/4 BUCKLING LOAD

Figure 4.7

N_ϕ vs θ

1/4 BUCKLING LOAD

TOROID PROGRAM

BOGGER

CIRCUMF. RESULTANT (lbs)

(DEGREES)

Figure 4.8

M_θ vs θ (BOSOR4)
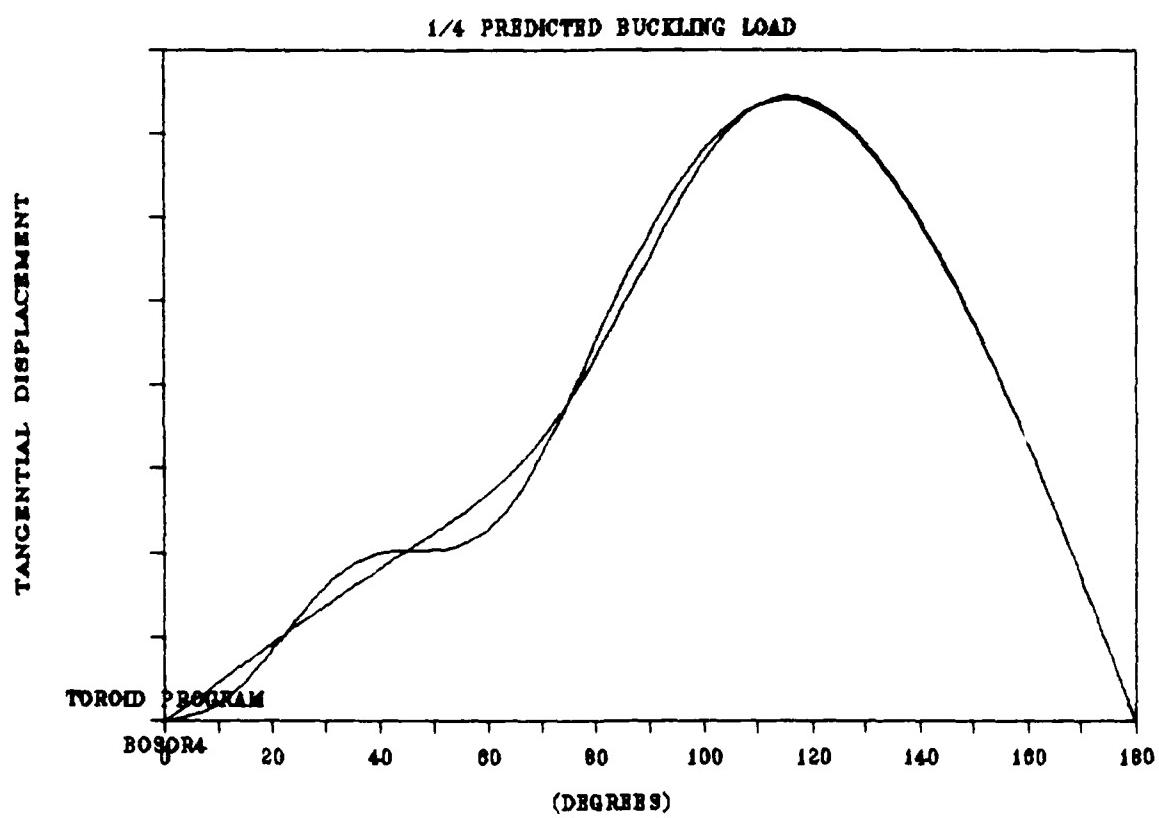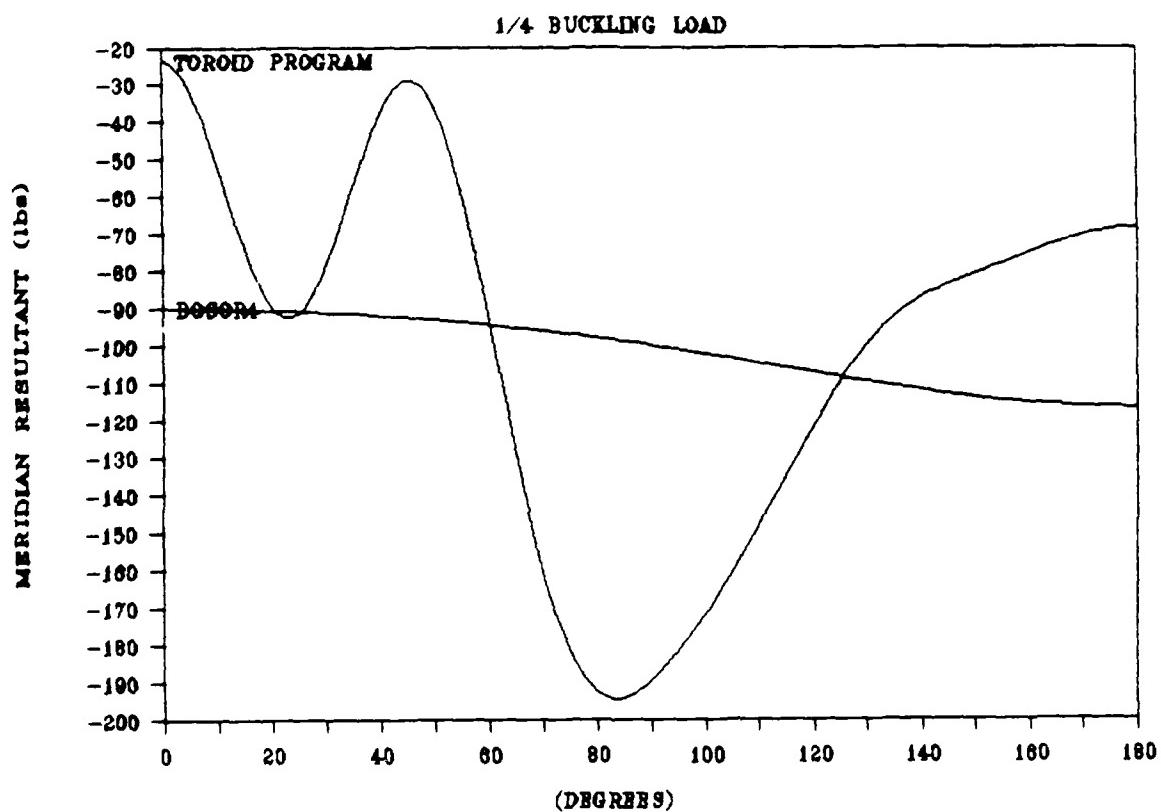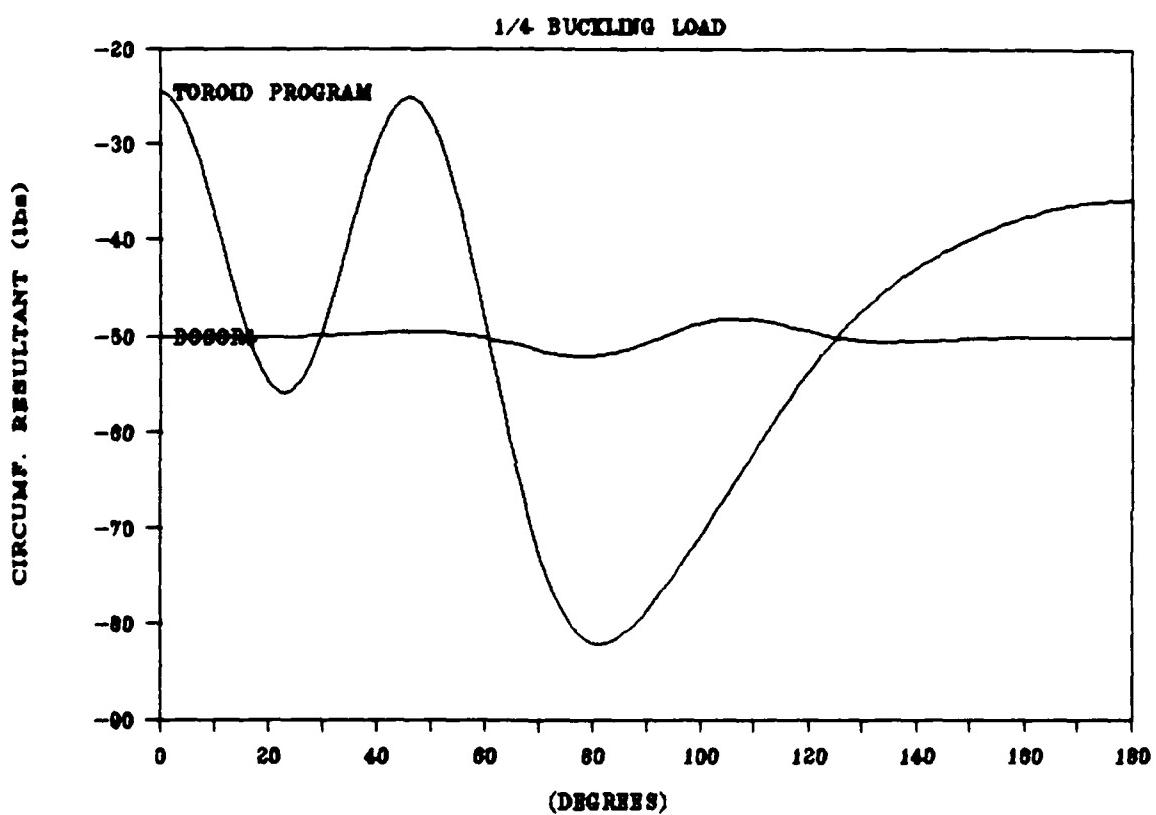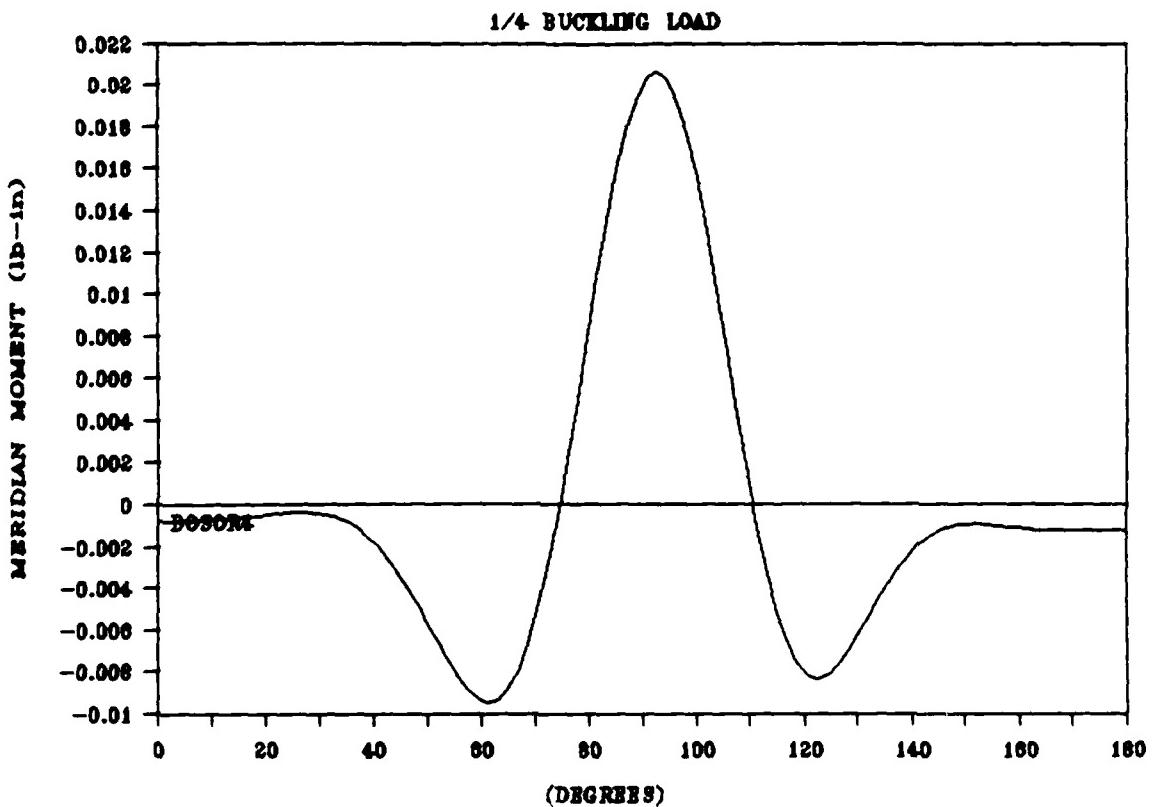
Figure 4.9

M_θ vs θ (Toroid Program)

Figure 4.10

M_y vs θ (BOSOR4)

1/4 BUCKLING LOAD
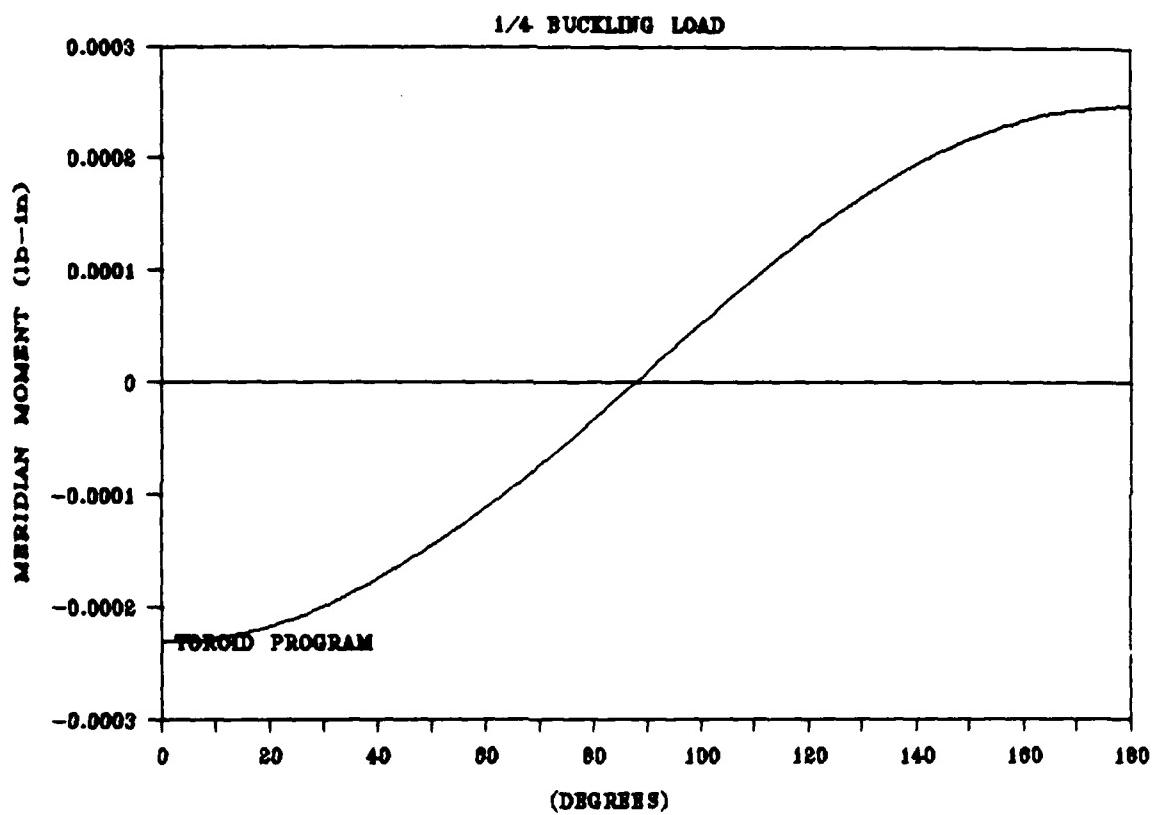
Plot: CIRCUMF. MOMENT (lb-in) vs (DEGREES). Curve labeled BOSOR4.

Figure 4.11

M_ϕ vs θ (Toroid Program)

1/4 BUCKLING LOAD
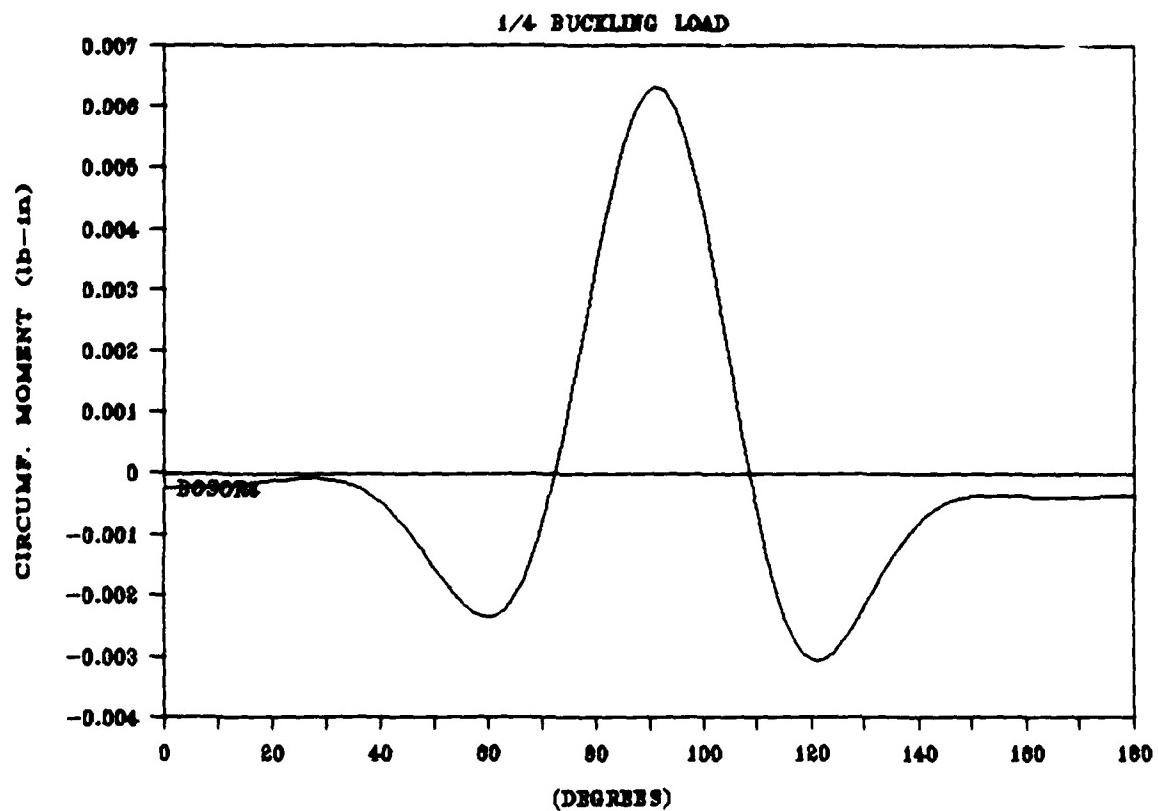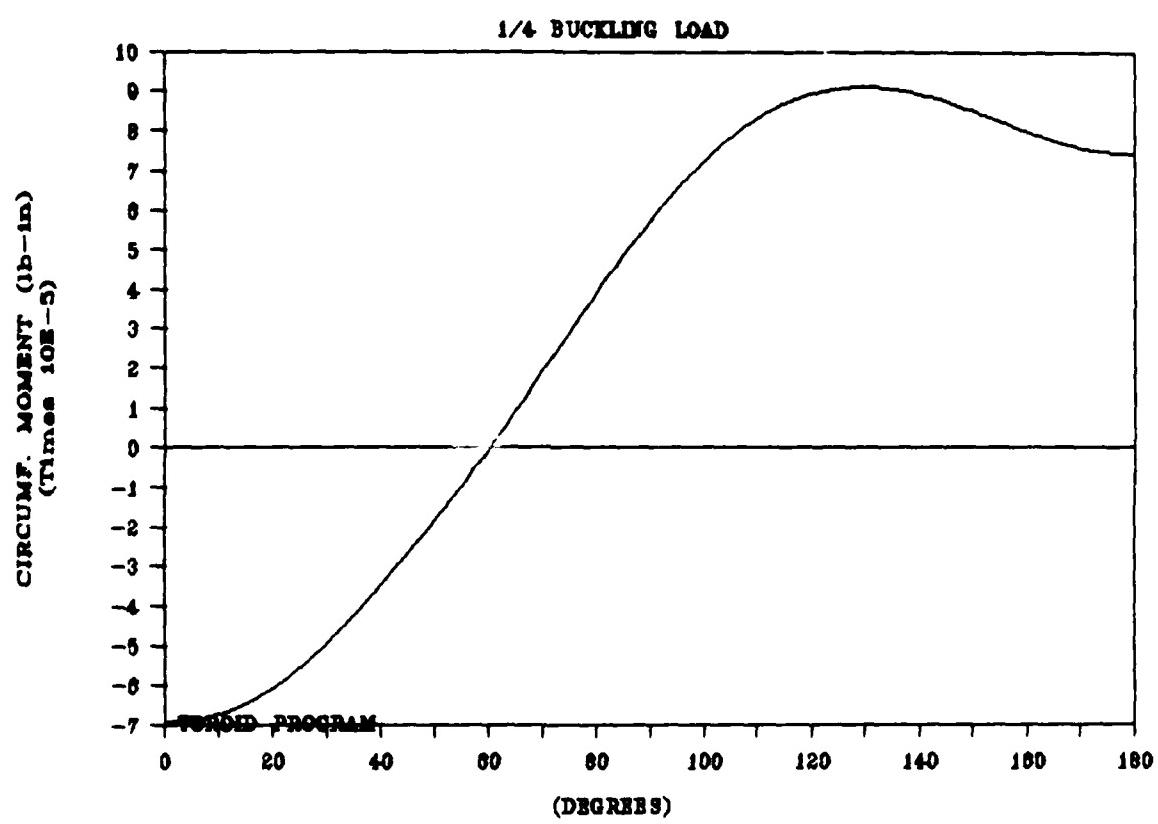
CHAPTER 5

CONCLUSIONS

As was shown in Chapter 4, using the assumed functions for the displacements in the energy method produced results for the displacements that were consistent with those obtained from BOSOR4. However, it was also shown that the "Toroid Program" was very poor at predicting the forces and moments of the toroid.

The differences can be explained by looking at what the assumed functions used for the displacements represents. The assumed functions for u and v are fairly good models of the displacements but were not modeled to meet the slope or curvature of the displacements except at the end points, where $\theta = 0$ and $\theta = \pi$. This resulted in good predictions for u and v but inaccurate predictions for u', u'' and v', all of which are contained in the expressions for the forces and moments.

Some thought has been given to what other assumed functions might be used to model the displacements that would more accurately predict the forces and moments. It is believed that a complete Fourier series representation, instead of just an odd or even series, might better predict the slope and curvature of the displacements and therefore the forces and moments. To ensure that the complete Fourier series meets the boundary conditions, Lagrange multipliers would be required to describe the constraint energy (See section 4.2.3). This is an area that is left for further investigation. If a complete Fourier Series

representation does not adequately predict the forces and moments the next step would be to write some kind of finite element or finite difference program to solve this problem.

Since the program written does model the displacements and produces results that are consistent with those obtained from BOSOR4, the "Toroid Program" can be used as a preliminary design tool without having to resort to a very complex and expensive computer code such as BOSOR4. Unfortunately the "Toroid Program" is restricted to analyzing only toroids under hydrostatic loading. Complexities in describing the geometry of a general shell of revolution will quickly lead the designer to the more complex computer codes like BOSOR4.

It is also believed that the program written could be modified to incorporate nonlinear terms and then be used to predict buckling loads. This is again an area that will be left for further investigation.

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APPENDIX A.

CURVE FITTING PROGRAMS

The programs listed were used in conjunction with the out put from BOSOR4 to determine the coefficients of the assumed Fourier Series. Both programs are written in Fortran 77.

Program 1 determines ten coefficients for an odd Fourier Series that is symmetric about θ equal to π, to fit the data from BOSOR4 for the normal displacement of the shell meridian.

Program 2 determines nine coefficients for an even Fourier Series that is anti-symmetric about θ equal to π, to fit the data from BOSOR4 for the tangential displacement of the shell meridian.

PROGRAM 1 LISTING:

```
C   CURVE FITTING PROGRAM
C   LEAST SQUARES CURVE FITTING
C234567
      INTEGER I,J,M,N
      REAL X(100),Y(100),F(100,10),FT(10,100),A(10,11),B(10)
      REAL F1,F2,F3,F4,F5,F6,F7,F8,F9,F10,C(10)
      EXTERNAL F1,F2,F3,F4,F5,F6,F7,F8,F9,F10
      N = 100
      M = 10
C   READ X-Y VALUES OF DATA POINTS
      DO 10 I=1,N
          READ (UNIT = 2,FMT = *) X(I)
          READ (UNIT = 3,FMT = *) Y(I)
```

```
10    CONTINUE
C  GENERATE THE F MATRIX
      DO 20 I=1,N
          F(I,1) = F1(X(I))
          F(I,2) = F2(X(I))
          F(I,3) = F3(X(I))
          F(I,4) = F4(X(I))
          F(I,5) = F5(X(I))
          F(I,6) = F6(X(I))
          F(I,7) = F7(X(I))
          F(I,8) = F8(X(I))
          F(I,9) = F9(X(I))
          F(I,10) = F10(X(I))
20    CONTINUE
C  GENERATE THE TRANSPOSE OF THE F MATRIX
      DO 30 I=1,N
      DO 30 J=1,M
          FT(J,I) = F(I,J)
30    CONTINUE
C  DETERMINE COEFFICIENT MATRIX A OF SIMULTANEOUS
C  EQUATION SYSTEM
      CALL MATMPY(FT,F,A,M,N,M)
C  DETERMINE COLUMN OF CONSTANTS FOR SIMULTANEOUS
C  EQUATION SYSTEM
      CALL MATMPY(FT,Y,B,M,N,1)
      DO 40 I=1,M
      A(I,M+1) = B(I)
40    CONTINUE
C  DETERMINE A(n) VALUES BY SOLVING SIMULTANEOUS EQUATIONS
C  USING CHOLESKY METHOD
      MP1 = M + 1
      CALL CHLSKY(A,M,MP1,C)
C  WRITE OUT THE A(n) VALUES
```

```fortran
      WRITE (UNIT = 4,FMT = *)(I,C(I),I=1,M)
      END
C
C   DETERMINES MATRIX C AS PRODUCT OF A AND B MATRICES
      SUBROUTINE MATMPY(A,B,C,M,N,L)
      REAL A(M,N),B(N,L),C(M,L)
      DO 10 I=1,M
      DO 10 J=1,L
      C(I,J) = 0.
      DO 10 K=1,N
      C(I,J) = C(I,J)+A(I,K)*B(K,J)
 10   CONTINUE
      END
C   DEFINE THE FUNCTIONS
C
C   DEFINE THE A(0) FUNCTION
      REAL FUNCTION F1(X)
      REAL X
      F1 = 1.0
      END
C   DEFINE THE A(1) FUNCTION
      REAL FUNCTION F2(X)
      REAL X
      F2 = COS(X)
      END
C   DEFINE THE A(2) FUNCTION
      REAL FUNCTION F3(X)
      REAL X
      F3 = COS(2.*X)
      END
C   DEFINE THE A(3) FUNCTION
      REAL FUNCTION F4(X)
      REAL X
```

```fortran
      F4 = COS(3.*X)
      END
C  DEFINE THE A(4) FUNCTION
      REAL FUNCTION F5(X)
      REAL X
      F5 = COS(4.*X)
      END
C  DEFINE THE A(5) FUNCTION
      REAL FUNCTION F6(X)
      REAL X
      F6 = COS(5.*X)
      END
C  DEFINE THE A(6) FUNCTION
      REAL FUNCTION F7(X)
      REAL X
      F7 = COS(6.*X)
      END
C  DEFINE THE A(7) FUNCTION
      REAL FUNCTION F8(X)
      REAL X
      F8 = COS(7.*X)
      END
C  DEFINE THE A(8) FUNCTION
      REAL FUNCTION F9(X)
      REAL X
      F9 = COS(8.*X)
      END
C  DEFINE THE A(9) FUNCTION
      REAL FUNCTION F10(X)
      REAL X
      F10 = COS(9.*X)
      END
C
```

```fortran
      SUBROUTINE CHLSKY(A,N,M,X)
      REAL A(10,11),X(10)
C  CALCULATE FIRST ROW OF UPPER UNIT TRIANGULAR MATRIX
      DO 10 J=2,M
      A(1,J) = A(1,J)/A(1,1)
 10   CONTINUE
C  CALCULATE OTHER ELEMENTS OF U AND L MATRICES
      DO 60 I=2,N
      J = I
      DO 30 II=J,N
      SUM = 0.
      JM1 = J-1
      DO 20 K=1,JM1
      SUM = SUM+A(II,K)*A(K,J)
 20   CONTINUE
      A(II,J) = A(II,J)-SUM
 30   CONTINUE
      IP1 = I+1
      DO 50 JJ=IP1,M
      SUM = 0.
      IM1 = I-1
      DO 40 K=1,IM1
      SUM = SUM+A(I,K)*A(K,JJ)
 40   CONTINUE
      A(I,JJ) = (A(I,JJ)-SUM)/A(I,I)
 50   CONTINUE
 60   CONTINUE
C  SOLVE FOR X(I) BY BACK SUBSTITUTION
      X(N) = A(N,N+1)
      L = N-1
      DO 80 NN=1,L
      SUM = 0.
      I = N-NN
```

54

```
         IP1 = I+1
         DO 70 J=IP1,N
         SUM = SUM+A(I,J)*X(J)
70    CONTINUE
         X(I) = A(I,M)-SUM
80    CONTINUE
         END
```

PROGRAM 2 LISTING:

```
C  CURVE FITTING
C  LEAST SQUARES CURVE FITTING
C234567
      INTEGER I,J,M,N
      REAL X(100),Y(100),F(100,9),FT(9,100),A(9,10),B(9)
      REAL F1,F2,F3,F4,F5,F6,F7,F8,F9,C(9)
      EXTERNAL F1,F2,F3,F4,F5,F6,F7,F8,F9
      N - 100
      M - 9
C  READ X-Y VALUES OF DATA POINTS
      DO 10 I-1,N
          READ (UNIT - 2,FMT - *) X(I)
          READ (UNIT - 5,FMT - *) Y(I)
 10   CONTINUE
C  GENERATE THE F MATRIX
      DO 20 I-1,N
          F(I,1) - F1(X(I))
          F(I,2) - F2(X(I))
          F(I,3) - F3(X(I))
          F(I,4) - F4(X(I))
          F(I,5) - F5(X(I))
          F(I,6) - F6(X(I))
          F(I,7) - F7(X(I))
          F(I,8) - F8(X(I))
          F(I,9) - F9(X(I))
 20   CONTINUE
C  GENERATE THE TRANSPOSE OF THE F MATRIX
      DO 30 I-1,N
      DO 30 J-1,M
          FT(J,I) - F(I,J)
 30   CONTINUE
```

56

```fortran
C   DETERMINE COEFFICIENT MATRIX A OF SIMULTANEOUS
C   EQUATION SYSTEM
      CALL MATMPY(FT,F,A,M,N,M)
C   DETERMINE COLUMN OF CONSTANTS FOR SIMULTANEOUS
C   EQUATION SYSTEM
      CALL MATMPY(FT,Y,B,M,N,1)
      DO 40 I=1,M
      A(I,M+1) = B(I)
 40   CONTINUE
C    DETERMINE B(n) VALUES BY SOLVING SIMULTANEOUS EQUATIONS
C   USING CHOLESKY METHOD
      MP1 = M + 1
      CALL CHLSKY(A,M,MP1,C)
C   WRITE OUT THE B(n) VALUES
      WRITE (UNIT = 4,FMT = *)(I,C(I),I=1,M)
      END
C
C   DETERMINES MATRIX C AS PRODUCT OF A AND B MATRICES
      SUBROUTINE MATMPY(A,B,C,M,N,L)
      REAL A(M,N),B(N,L),C(M,L)
      DO 10 I=1,M
      DO 10 J=1,L
      C(I,J) = 0.
      DO 10 K=1,N
      C(I,J) = C(I,J)+A(I,K)*B(K,J)
 10   CONTINUE
      END
C
C   DEFINE THE FUNCTIONS
C
C   DEFINE THE B(1) FUNCTION
      REAL FUNCTION F1(X)
      REAL X
```

57

```fortran
      F1 = SIN(X)
      END
C  DEFINE THE B(2) FUNCTION
      REAL FUNCTION F2(X)
      REAL X
      F2 = SIN(2.*X)
      END
C  DEFINE THE B(3) FUNCTION
      REAL FUNCTION F3(X)
      REAL X
      F3 = SIN(3.*X)
      END
C  DEFINE THE B(4) FUNCTION
      REAL FUNCTION F4(X)
      REAL X
      F4 = SIN(4.*X)
      END
C  DEFINE THE B(5) FUNCTION
      REAL FUNCTION F5(X)
      REAL X
      F5 = SIN(5.*X)
      END
C  DEFINE THE B(6) FUNCTION
      REAL FUNCTION F6(X)
      REAL X
      F6 = SIN(6.*X)
      END
C  DEFINE THE B(7) FUNCTION
      REAL FUNCTION F7(X)
      REAL X
      F7 = SIN(7.*X)
      END
C  DEFINE THE B(8) FUNCTION
```

```fortran
      REAL FUNCTION F8(X)
      REAL X
      F8 - SIN(8.*X)
      END
C  DEFINE THE B(9) FUNCTION
      REAL FUNCTION F9(X)
      REAL X
      F9 - SIN(9.*X)
      END
C
      SUBROUTINE CHLSKY(A,N,M,X)
      REAL A(9,10),X(9)
C  CALCULATE FIRST ROW OF UPPER UNIT TRIANGULAR MATRIX
      DO 10 J=2,M
      A(1,J) - A(1,J)/A(1,1)
 10   CONTINUE
C  CALCULATE OTHER ELEMENTS OF U AND L MATRICES
      DO 60 I=2,N
      J - I
      DO 30 II=J,N
      SUM - 0.
      JM1 - J-1
      DO 20 K-1,JM1
      SUM - SUM+A(II,K)*A(K,J)
 20   CONTINUE
      A(II,J) - A(II,J)-SUM
 30   CONTINUE
      IP1 - I+1
      DO 50 JJ=IP1,M
      SUM - 0.
      IM1 - I-1
      DO 40 K-1,IM1
      SUM - SUM+A(I,K)*A(K,JJ)
```

```fortran
 40     CONTINUE
        A(I,JJ) = (A(I,JJ)-SUM)/A(I,I)
 50     CONTINUE
 60     CONTINUE
C   SOLVE FOR X(I) BY BACK SUBSTITUTION
        X(N) = A(N,N+1)
        L = N-1
        DO 80 NN=1,L
        SUM = 0.
        I = N-NN
        IP1 = I+1
        DO 70 J=IP1,N
        SUM = SUM+A(I,J)*X(J)
 70     CONTINUE
        X(I) = A(I,M)-SUM
 80     CONTINUE
        END
```

APPENDIX B.

TOROID PROGRAM

This program is set up to determine the ten coefficients of the even Fourier Series for the normal displacement, (u), of the toroid and the nine coefficients of the odd Fourier Series for the tangential displacement, (v), of the toroid.

To determine the shape and magnitude of the normal and tangential displacement of the toroid the following equations are used:

$$u = \sum_{n=0}^{9} a_n \cos(n\theta)$$

$$v = \sum_{n=1}^{9} b_n \sin(n\theta)$$

The program was written in Fortran 77. The subroutines used were adopted from those presented by Dyck, Lawson and Smith (1984).

PROGRAM LISTING:

```
C  TOROID PROGRAM
      INTEGER I,N
      REAL F(19,19),D(19),C(19)
      REAL E,V,RB,RL,H,P,A1,A2,A3,A
      LOGICAL ERROR
      EXTERNAL FA,FA1,FB,FB1,SIMP
      PRINT *,'This Program will calculate the coefficients'
      PRINT *,'of the Fourier Series for the Normal and'
```

```fortran
      PRINT *,'Tangential displacements of a thin, circular'
      PRINT *,'toroidal shell when loaded by uniform'
      PRINT *,'external hydrostatic pressure.'
      PRINT *,' '
      PRINT *,'Enter Youngs Modulus, E(psi) -'
      READ *,E
      PRINT *,'Enter Poissons Ratio, V -'
      READ *,V
      PRINT *,'Enter Toroid radius, R(in) -'
      READ *,RB
      PRINT *,'Enter Circular Section radius, r(in) -'
      READ *,RL
      PRINT *,'Enter Shell thickness, h(in) -'
      READ *,H
      PRINT *,'Enter hydrostatic pressure, P(psi) -'
      READ *,P
      PI - 4.0*ATAN(1.0)
      N - 19
      A - RB/RL
      A1=2.0*PI*E*H/(1.-V**2)
      A2=-2.0*PI*P*RL**2
      A3=(-PI*E*(H**3))/(6.0*(RL**2)*(1.0-V**2))
C  GENERATE THE D COLUMN
      D(1)=A2*PI*A
      D(2)=A2*PI/2.0
      DO 30 I=3,N
         D(I)=0.0
 30   CONTINUE
C  GENERATE THE F MATRIX
      DO 40 I=1,N
         PRINT *,'INTEGRATING MATRIX COLUMN',I
         F(1,I) - A1*SIMP(FA,PI,V,A,I,0.0)+A3*SIMP(FA1,PI,V,A,I,0.0)
         F(2,I) - A1*SIMP(FA,PI,V,A,I,1.0)+A3*SIMP(FA1,PI,V,A,I,1.0)
```

```
      F(3,I) = A1*SIMP(FA,PI,V,A,I,2.0)+A3*SIMP(FA1,PI,V,A,I,2.0)

      F(4,I) = A1*SIMP(FA,PI,V,A,I,3.0)+A3*SIMP(FA1,PI,V,A,I,3.0)

      F(5,I) = A1*SIMP(FA,PI,V,A,I,4.0)+A3*SIMP(FA1,PI,V,A,I,4.0)

      F(6,I) = A1*SIMP(FA,PI,V,A,I,5.0)+A3*SIMP(FA1,PI,V,A,I,5.0)

      F(7,I) = A1*SIMP(FA,PI,V,A,I,6.0)+A3*SIMP(FA1,PI,V,A,I,6.0)

      F(8,I) = A1*SIMP(FA,PI,V,A,I,7.0)+A3*SIMP(FA1,PI,V,A,I,7.0)

      F(9,I) = A1*SIMP(FA,PI,V,A,I,8.0)+A3*SIMP(FA1,PI,V,A,I,8.0)

      F(10,I) = A1*SIMP(FA,PI,V,A,I,9.0)+A3*SIMP(FA1,PI,V,A,I,9.0)

      F(11,I) = A1*SIMP(FB,PI,V,A,I,1.0)+A3*SIMP(FB1,PI,V,A,I,1.0)

      F(12,I) = A1*SIMP(FB,PI,V,A,I,2.0)+A3*SIMP(FB1,PI,V,A,I,2.0)

      F(13,I) = A1*SIMP(FB,PI,V,A,I,3.0)+A3*SIMP(FB1,PI,V,A,I,3.0)

      F(14,I) = A1*SIMP(FB,PI,V,A,I,4.0)+A3*SIMP(FB1,PI,V,A,I,4.0)

      F(15,I) = A1*SIMP(FB,PI,V,A,I,5.0)+A3*SIMP(FB1,PI,V,A,I,5.0)

      F(16,I) = A1*SIMP(FB,PI,V,A,I,6.0)+A3*SIMP(FB1,PI,V,A,I,6.0)

      F(17,I) = A1*SIMP(FB,PI,V,A,I,7.0)+A3*SIMP(FB1,PI,V,A,I,7.0)

      F(18,I) = A1*SIMP(FB,PI,V,A,I,8.0)+A3*SIMP(FB1,PI,V,A,I,8.0)

      F(19,I) = A1*SIMP(FB,PI,V,A,I,9.0)+A3*SIMP(FB1,PI,V,A,I,9.0)
   40 CONTINUE
C  CONVERT THE LINEAR SYSTEM TO A TRIANGULAR SYSTEM
      CALL GAUSS(F,D,N,ERROR)
      IF (ERROR) THEN
         PRINT *,'MATRIX GENERATED IS SINGULAR,'
         PRINT *,'SOLUTION IS NOT POSSIBLE.'
      ELSE
C  SOLVE THE TRIANGULAR LINEAR SYSTEM
      CALL BSOLVE(F,D,N,C)
      PRINT *,'THE FOLLOWING ARE THE 10 COEFFICIENTS FOR THE'
      PRINT *,'NORMAL DISPLACEMENT OF THE TOROID.'
      PRINT *,' '
      PRINT *,'USE IN THE SERIES; An*COS(n*theta/r)'
      PRINT *,' '
      PRINT *,'          n          An          n
     +          An'
```

```fortran
      PRINT *,(I-1,C(I),I=1,10)
      PRINT *, ' '
      PRINT *,'THE FOLLOWING ARE THE 9 COEFFICIENTS FOR THE'
      PRINT *,'TANGENTIAL DISPLACEMENT OF THE TOROID.'
      PRINT *,' '
      PRINT *,'USE IN THE SERIES; Bn*SIN(n*theta/r)'
      PRINT *,' '
      PRINT *,'           n              Bn            n
     +          Bn'
      PRINT *,(I-10,C(I),I=11,19)
      ENDIF
      END
C
C  THIS SUBPROGRAM FUNCTION INTEGRATES THE MATRIX FUNCTIONS
C  DEFINED BY USING SIMPSON'S RULE AS AN APPROXIMATION
      REAL FUNCTION SIMP(F,PI,V,A,N,Z)
      REAL PI,F,H,SUMEVN,SUMODD,X,B
      INTEGER I
      EXTERNAL F
      B=0.0
      H=PI/100.
      SUMEVN=0.0
      SUMODD=F(A,V,N,Z,H)
      DO 100 I=1,49
         X=2.*I*H
         SUMEVN=SUMEVN+F(A,V,N,Z,X)
         SUMODD=SUMODD+F(A,V,N,Z,X+H)
  100 CONTINUE
      SIMP=(H/3.)*(F(A,V,N,Z,B)+4.*SUMODD+2.*SUMEVN+F(A,V,N,Z,PI))
      RETURN
      END
C
C
```

```fortran
C   DEFINE THE MATRIX FUNCTIONS
C   CALCULATE ROWS 1 THROUGH 10
C   FUNCTION FA REPRESENTS MEMBRANE ENERGY
      REAL FUNCTION  FA(A,V,I,Z,X)
      REAL Q
      IF (I .LT. 11) THEN
         Q=I-1
         FA=(A+COS(X))*COS(Q*X)*COS(Z*X)+2.0*V*COS(X)*COS(Q*X)*
     +      COS(Z*X)+(COS(X)**2)*COS(Q*X)*COS(Z*X)/(A+COS(X))
      ELSE
         Q=I-10
         FA=(A+COS(X))*Q*COS(Q*X)*COS(Z*X)+V*(COS(X)*Q*COS(Q
     +      *X)*COS(Z*X)-SIN(X)*SIN(Q*X)*COS(Z*X))-COS(X)*SI
     +      N(X)*SIN(Q*X)*COS(Z*X)/(A+COS(X))
      ENDIF
      END
C   FUNCTION FA1 REPRESENTS BENDING ENERGY
      REAL FUNCTION  FA1(A,V,I,Z,X)
      REAL Q
      IF (I .LT. 11) THEN
         Q=I-1
         FA1=(A+COS(X))*((Q*Z)**2)*COS(Q*X)*COS(Z*X)-V*SIN(X
     +      )*(Q*(Z**2)*SIN(Q*X)*COS(Z*X)+(Q**2)*Z*COS(Q*X)*
     +      SIN(Z*X))+(SIN(X)**2)*Q*Z*SIN(Q*X)*SIN(Z*X)/(A+C
     +      OS(X))
      ELSE
         Q=I-10
         FA1=(A+COS(X))*Q*(Z**2)*COS(Q*X)*COS(Z*X)-V*SIN(X)*
     +      (Q*Z*COS(Q*X)*SIN(Z*X)+(Z**2)*SIN(Q*X)*COS(Z*X))
     +      +(SIN(X)**2)*Z*SIN(Q*X)*SIN(Z*X)/(A+COS(X))
      ENDIF
      END
C
```

```fortran
C   CALCULATE ROWS 11 THROUGH 19
C   FUNCTION FB REPRESENTS MEMBRANE ENERGY
      REAL FUNCTION  FB(A,V,I,Z,X)
      REAL Q
      IF (I .LT. 11) THEN
         Q=I-1
         FB=(A+COS(X))*Z*COS(Q*X)*COS(Z*X)+V*(COS(X)*Z*COS(Q
     +       *X)*COS(Z*X)-SIN(X)*COS(Q*X)*SIN(Z*X))-COS(X)*SI
     +       N(X)*COS(Q*X)*SIN(Z*X)/(A+COS(X))
      ELSE
         Q=I-10
         FB=(A+COS(X))*Q*Z*COS(Q*X)*COS(Z*X)-V*SIN(X)*(Z*SIN
     +       (Q*X)*COS(Z*X)+Q*COS(Q*X)*SIN(Z*X))+(SIN(X)**2)*
     +       SIN(Q*X)*SIN(Z*X)/(A+COS(X))
      ENDIF
      END
C   FUNCTION FB1 REPRESENTS BENDING ENERGY
      REAL FUNCTION  FB1(A,V,I,Z,X)
      REAL Q
      IF (I .LT. 11) THEN
         Q=I-1
         FB1=(A+COS(X))*Z*(Q**2)*COS(Q*X)*COS(Z*X)-V*SIN(X)*
     +       (Q*Z*SIN(Q*X)*COS(Z*X)+(Q**2)*COS(Q*X)*SIN(Z*X))
     +       +(SIN(X)**2)*Q*SIN(Q*X)*SIN(Z*X)/(A+COS(X))
      ELSE
         Q=I-10
         FB1=(A+COS(X))*Z*Q*COS(Q*X)*COS(Z*X)-V*SIN(X)*(Z*SI
     +       N(Q*X)*COS(Z*X)+Q*COS(Q*X)*SIN(Z*X))+(SIN(X)**2)
     +       *SIN(Q*X)*SIN(Z*X)/(A+COS(X))
      ENDIF
      END
C
C   GAUSSIAN ELIMINATION MODULE
```

```fortran
      SUBROUTINE GAUSS(A,B,M,ERROR)
C
      INTEGER M,I,J,K,INDEX
      REAL A(M,M),B(M),PIVOT,TEMP,RATIO
      LOGICAL ERROR
      ERROR=.FALSE.
      DO 100 I=1,M
      INDEX=I
      PIVOT=ABS(A(I,I))
      DO 200 J=I+1,M
         IF (ABS(A(J,I)) .GT. PIVOT) THEN
            PIVOT=ABS(A(J,I))
            INDEX=J
         ENDIF
200   CONTINUE
      IF (INDEX .GT. I) THEN
         DO 400 K=I,M
            TEMP=A(I,K)
            A(I,K)=A(INDEX,K)
            A(INDEX,K)=TEMP
400      CONTINUE
         TEMP=B(I)
         B(I)=B(INDEX)
         B(INDEX)=TEMP
      ENDIF
      IF (PIVOT .EQ. 0.0) THEN
         ERROR = .TRUE.
      ELSE
         DO 300 J=I+1,M
            RATIO=A(J,I)/A(I,I)
            DO 500 K=I+1,M
               A(J,K)=A(J,K)-A(I,K)*RATIO
500         CONTINUE
            B(J)=B(J)-B(I)*RATIO
```

```fortran
300     CONTINUE
      ENDIF
100   CONTINUE
      END
C
C  SUBROUTINE TO SOLVE TRIANGULAR LINEAR SYSTEM
      SUBROUTINE BSOLVE(A,B,M,Z)
      INTEGER M,I,J
      REAL A(M,M),B(M),Z(M),SUM
      DO 200 I-M,1,-1
         SUM-B(I)
         DO 100 J-I+1,M
            SUM=SUM-A(I,J)*Z(J)
100      CONTINUE
         Z(I)-SUM/A(I,I)
200   CONTINUE
      END
```